Structure of Singular Sets Local to Cylindrical Singularities for Stationary Harmonic Maps and Mean Curvature Flows

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Abstract

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In this paper we prove structure results for the singular sets of stationary harmonic maps and mean curvature flows local to particular singularities. The original work is contained in Chapter 5 and Chapter 8. Chapters 1-5 are concerned with energy minimising maps and stationary harmonic maps. Chapters 6-8 are concerned with mean curvature flows and Brakke flows.

In the case of stationary harmonic maps we consider a singularity at which the spine dimension is maximal, and such that the weak tangent map is homotopically non-trivial, and has minimal density amongst singularities of maximal spine dimension. Local to such a singularity we show the singular set is a bi-Hölder continuous homeomorphism of the unit disk of dimension equal to the maximal spine dimension. A weak tangent map is translation invariant along a subspace, and invariant under dilations, so it completely defined by its values on a sphere. Such a map is said to be homotopically non-trivial if the mapping of a sphere into some target manifold cannot be deformed by a homotopy to a constant map.

For an n-dimensional mean curvature flow we consider a singularity at which we can find a shrinking cylinder as a tangent flow, that collapses on an $(n-1)$-dimensional plane. Local to such a singularity we show that all singularities have such a cylindrical tangent, or else have lower Gaussian density than that of the shrinking cylinder. The subset of cylindrical singularities can be shown to be contained in a finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds. In the case that the mean curvature flow arises from elliptic regularisation we can show that all singularities local to a cylindrical singularity with $(n-1)$-dimensional spine are either cylindrical singularities with $(n-1)$-dimensional spine, or contained in a parabolic Hausdorff $(n-2)$-dimensional set.

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Contents

Chapter 1

Introduction - Stationary Harmonic Maps

1.1 Background

Stationary harmonic maps arise as a natural generalisation of energy minimising maps. Given an open set $\Omega \subset \mathbb{R}^N$ and a smooth target manifold N an energy minimising map is a map $u : \Omega \to N$ with the property that the energy $\mathcal{E}_B(u) = \|Du\|_{L^2(B)}$ satisfies $\mathcal{E}_B(u) \leq \mathcal{E}_B(w)$ for every ball B and $w \in W^{1,2}(B; N)$ with $w = u$ on ∂B. A stationary harmonic map $u \in W^{1,2}_{loc}(\Omega; N)$ is a map for which the energy functional $\mathcal{E}_B(u)$ has first variation equal to 0 on any ball $B \subset\subset \Omega$. Clearly all energy minimising maps are stationary harmonic. By taking the first variation with respect to two different types of variation it can be shown that stationary harmonic maps satisfy two non-linear elliptic partial differential equations.

Both energy minimising and stationary harmonic maps can admit singularities. A monotonicity formula for the energy density ratios $|B|^{-1}\mathcal{E}_B(u)$ follows from the first variation of energy being 0. From this it follows that we can define an energy density $\Theta_u(x)$ by the limit of the energy density ratios on a sequence of shrinking balls centred on x. Energy minimising maps admit energy minimising tangent maps by monotonicity and a compactness theorem. For stationary harmonic maps the compactness is not so simple and we must extend to a class of measures that can be obtained as the weak limits of the energy measures $|Du|^2 \mathrm{d}x$ of stationary harmonic maps. This class of measures was also studied by Lin [Lin99] and many properties of stationary harmonic maps extend naturally to this class of measures. Through analysis of the tangents and the energy density function we can study the singularities of energy minimising and stationary harmonic maps. In particular the Schoen-Uhlenbeck theorem [SU82] for energy-minimising maps and the Bethuel regularity theorem [Bet93] for stationary harmonic maps can be thought of as saying the map is regular near to any point that admits a constant function as a tangent. Further results such as the stratification and dimension reduction of the singular set allow us to study the finer structure of the singular set. The main result of this paper for stationary harmonic maps can be proved by making use of these results, and the Reifenberg topological disk theorem [Rei60].

1.2 Main Results

The main result of this paper for stationary harmonic maps is about the structure of the singular set local to a singularity with particular properties. We assume that at this fixed singularity there is a tangent with maximal spine dimension, the slices of the weak tangent map along the spine cannot be deformed to a constant by a homotopy, and that the energy density at this singularity is minimal among all possible tangents with maximal spine dimension. By a tangent we refer to the tangent map of an energy minimising map, or the weak tangent measure of a stationary harmonic map. In both cases the tangent has a spine, a subspace of $\mathbb{R}^n$ along which the tangent is translation and dilation invariant. We say a singularity is cylindrical if there is a tangent with maximal spine dimension among all tangents at singularities. This dimension is bounded above by $n-3$ for energy minimising maps on n-dimensional domains, and $n-2$ for stationary harmonic maps on n-dimensional domains. In the case of stationary harmonic maps we can also identify a weak tangent map associated to the tangent measure, however this map is only weakly harmonic, and not stationary harmonic. We say a tangent map is homotopically non-trivial when the slices of the tangent map along the spine, restricted to a sphere, cannot be deformed to a constant by homotopy. We call the minimal density among cylindrical singularities the minimal cylindrical density. The following paragraphs give some more detail on these properties.

We fix some smooth target manifold N, which can be thought of as embedded in some Euclidean space. The spine of a tangent map $\phi : \mathbb{R}^n \to N$ is the collection of points $x \in \mathbb{R}^n$ for which the energy density satisfies $\Theta_\phi(x) = \Theta_\phi(0)$. It is well known this forms a linear subspace of $\mathcal{S}(\phi) \subset \mathbb{R}^n$, and ϕ is translation-invariant along for translations by vectors in $\mathcal{S}(\phi)$. By dimension reduction arguments and regularity results we have that $\dim(\mathcal{S}(\phi)) \leq n-3$ for tangents to energy minimising maps. In the case of stationary harmonic maps we instead need to work with a tangent measure, however this also has a spine defined in a similar way, and it can be shown that this spine of the measure is a linear subspace of dimension at most $n-2$. Given a fixed domain and target we can then find some fixed dimension $d \leq n-2$ such that all spines have dimension at most d. We will then say a tangent is cylindrical if it attains this dimension.

Given a map $\phi : \mathbb{R}^n \to N$ we can take a slice map $\phi\lfloor_{V^\perp,x} : V^\perp \to N$ by $\phi\lfloor_{V^\perp,x}(y) = \phi(x+y)$ for each $x \in V$ where $V \subset \mathbb{R}^n$ is a linear subspace. When ϕ is a tangent map and V is the spine of either ϕ or some associated tangent measure, we actually have that $\phi\lfloor_{V^\perp,x}$ is independent of $x \in V$ by translation invariance. We say ϕ is homotopically non-trivial if there is no homotopy between $\phi\lfloor V^\perp, 0$, restricted to the unit sphere, and a constant map. For particular choices of target manifold N the homotopically non-trivial property of the tangent is automatic.

The main result is Theorem 5.5.10, which is stated for a class of measures attained as weak limits of energy measures of stationary harmonic maps. In the case of energy minimising maps this theorem is as follows.

Theorem 1.2.1 (The Structure Theorem for Energy Minimising Maps)**.** *Let $N \subset \mathbb{R}^m$ denote a smooth manifold isometrically embedded in $\mathbb{R}^m$. Consider an energy-minimising map $u \in W^{1,2}(B_1; N)$ on the n-dimensional ball $B_1 \subset \mathbb{R}^n$, and suppose $x_0 \in \mathrm{Sing}(u)$. Let $d_u \leq n-3$ denote the maximal spine dimension of all homogeneous degree zero limit maps to u. Let $\alpha_u > 0$ denote the minimal density of all homogeneous degree zero limit maps to u with spine dimension d_u. Suppose $\Theta_u(x_0) = \alpha_u$ and there is a tangent map $\phi \in T_x u$ with d_u-dimensional spine $\mathcal{S}(\phi)$. Finally suppose the slice maps $\phi\lfloor_{\mathcal{S}(\phi)^\perp,0}$ restricted to $\partial B_{1/2}^{n-d_u} \subset \mathcal{S}(\phi)^\perp$ cannot be deformed to a constant map via a homotopy. Then for any $\beta \in (0,1)$ there is $\delta = \delta(u, x_0, N, n, \beta) > 0$ such that $\mathrm{Sing}(u) \cap B_\delta(x_0)$ can be mapped onto a d_u-dimensional ball by a bi-Hölder continuous map with exponent β.*

Many of these conditions can be easily verified in two particular cases, the targets $N = S^2$ and $N = S^3$. In the $N = S^2$ case this is due to the result of Brezis-Coron-Lieb [BCL86] which classifies the possible tangent maps to an energy minimising map from a 3-dimensional domain to S^2. All such tangent maps are rotations of the identity map on S^2, extended radially. These maps have degree ± 1, and so cannot be homotopically equivalent to a constant. This also implies that for an energy minimising map $u \in W^{1,2}(B_1, S^2)$, with $B_1 \subset \mathbb{R}^n$, that all tangent maps with $n-3$-dimensional spines have the same energy density. As such the above Theorem applies at any singular point in the top-dimensional stratum of the singular set of an energy minimising map onto S^2.

Corollary 1.2.2 (Energy minimising maps onto S^2)**.** *Let $u \in W^{1,2}_{loc}(B_1; S^2)$ denote an energy minimising map. We can find closed subsets $T, S \subset B_1$ such that* $\mathrm{Sing}(u) = S \cup T$*, and for any $\beta \in (0,1)$ the following properties hold.*

- $\dim_{\mathcal{H}}(T) \leq n - 4$
- *For each $x \in S$ there is $r = r(u, x, n, \beta) > 0$ such that*

$$\mathrm{Sing}(u) \cap B_r(x) = S \cap B_r(x).$$

- *For each $x \in S$, there is $r = r(u, x, n, \beta) > 0$ such that $S \cap \overline{B}_r(x)$ is the image of an $(n-3)$-dimensional disk under a bi-Hölder map with exponent $\beta > 0$*

In the case of the target $N = S^3$ it was shown by Schoen-Uhlenbeck [SU84] that there are no non-constant energy minimising tangent maps from $\mathbb{R}^3$ to S^3, implying $d_u \leq n - 4$ for any energy minimising $u \in W^{1,2}(B_1; S^3)$. The tangent maps with $(n-4)$-dimensional spine are classified by Nakajima [Nak06]. Here it is shown that all stable-stationary harmonic maps from B^4 to S^3 are isometries of the map $x \to x/|x|$. As such the assumptions of the above theorem are automatically satisfied for the top-dimensional singular stratum.

Corollary 1.2.3 (Energy minimising maps onto S^3)**.** *Let $u \in W^{1,2}_{loc}(B_1; S^3)$ denote an energy minimising map. We can find closed subsets $T, S \subset B_1$ such that* $\mathrm{Sing}(u) = S \cup T$*, and for any $\beta \in (0,1)$ the following properties hold. following properties.*

- $\dim_{\mathcal{H}}(T) \leq n - 5$
- *For each $x \in S$ there is $r = r(u, x, n, \beta) > 0$ such that*

$$\mathrm{Sing}(u) \cap B_r(x) = S \cap B_r(x).$$

- *For each $x \in S$, there is $r = r(u, x, n, \beta) > 0$ such that $S \cap \overline{B}_r(x)$ is the image of an $(n-4)$-dimensional disk under a bi-Hölder map with exponent $\beta > 0$*

1.3 Known Results

There are a number of results on the structure of singular sets to geometric variational problems. These are briefly discussed below and contrasted to Theorem 1.2.1, Corollary 1.2.2 and Corollary 1.2.3.

It was shown by Simon [Sim95] that the singular set of an energy minimising map is locally a union of rectifiable sets, of dimension equal to the maximal possible singular dimension of all energy-minimising homogeneous degree zero maps onto a fixed target N. Our structure result is slightly weaker in the sense that we do not get Lipschitz regularity, however it is stronger in that locally the singular set is equal to a single Hölder continuous sheet rather than multiple Lipschitz sheets. A rectifiability result for the singular set of a minimal submanifold was also proved by Simon [Sim93], in particular proving that a submanifold M sufficiently close to a cylinder C satisfies a decomposition result for the set of singularities $x \in \mathrm{Sing}(M)$ with density at least equal to the density of C at the origin. We will prove a similar decomposition result here for energy-minimising maps, and for a class of measures arising as the weak limit of energy measures of stationary harmonic maps.

Hardt-Lin [HL90] show that the singular set of an energy minimising map $u : B_1^4 \to S^2$ is contained in a Hölder continuous arc. Corollary 1.2.2 extends this to arbitrary dimensional domains.

In the case of stationary harmonic maps, Lin [Lin99] considers a class of measures μ obtained as weak limit of energy measures $|Du_i|^2 \mathrm{d}x$ for stationary harmonic maps $u_i : \mathbb{R}^n \to N$. It is shown that these measures have a generalised singular set $\Sigma(\mu)$ and that this is $(n-2)$-rectifiable. In fact this also follows from Preiss' theorem [Pre87], however the proof is shorter in the case of these weak limits of energy measures. Again our structure result aims to show that local to particular singularities this generalised singular set is the Hölder continuous image of a single disk.

A quantitative stratification of the singular set of both harmonic maps and minimal currents was shown by Cheeger and Naber [CN13]. Here the usual Hausdorff dimension bound on the singular set of an energy-minimizing map is extended to a Minkowski dimension bound. This is extended to mean curvature flows and harmonic map flow by Cheeger, Haslhofer and Naber in [CHN13] and [CHN15] respectively. For stationary harmonic maps Naber and Valtorta [NV17] prove volume estimates on neighbourhoods of the quantitative singular strata, and that the singular strata are rectifiable of dimension equal to the dimension of the stratum. Similar results are proved for stationary varifolds by Naber and Valtorta in [NV15]. The key difference between these results and Theorem 1.2.1 is that Theorem 1.2.1 gives conditions under which we can assure the singular set is a single bi-Hölder continuous disk, whereas rectifiability means the singular strata are unions of Lipschitz submanifolds. The trade off for to prove that the singular set is a single bi-Hölder disk is that we need strict conditions on the singularity, and we only get bi-Hölder regularity, not Lipschitz.

1.4 Outline of the Chapters

Chapter 2 outlines some of the notation and simple background results that will be useful. We will work with spaces of $W^{1,2}$ functions and Radon measures. Further we need to metrise the weak convergence in these spaces. For the structure result we wish to show that local to a homotopically non-trivial singularity with minimal cylindrical density the singular set has is bi-Hölder homeomorphic to a disk. This kind of structure for a general closed set follows from the Reifenberg topological disk theorem [Rei60]. The main condition of this theorem is the Reifenberg approximation condition that the closed set can be well approximated by planes at each point and scale. Details on the Reifenberg theorem are also covered in this chapter.

We introduce energy minimising and stationary harmonic maps in Chapter 3. This includes the monotonicity formula, the definition of the energy density, the regularity theorems of Schoen-Uhlenbeck [SU82] and Bethuel [Bet93], and analysis of the tangent maps in the case of energy minimising maps.

In Chapter 4 we study the class of measures extending stationary harmonic maps, and properties of this class. This class was studied previously by Lin [Lin99] and section 4 will recall the results from this paper that we need.

With this background material we can proceed in Chapter 5 to the main arguments. The are four key steps in the argument, these sections are outlined below.

To satisfy the Reifenberg approximation condition we need a collection of planes that approximate the singular set well according to Hausdorff distance. The spines of tangents are good planes to approximate the singular set, however only at scale 0. To achieve approximations at positive scales we define pseudo-tangent maps in section 5.2 which approximate a stationary harmonic map translated and dilated by some finite positive scale. These pseudo-tangents will be homogeneous degree zero, and as such have well defined spines. However it is not immediately clear that the spines are all of the same dimension, or that they behave well when changing the base point or the scale. Further these pseudo-tangents are only defined on a subset of the singular set.

To solve the issue of the dimension of the spines we prove a rigidity theorem in section 5.3. This shows that if a pseudo-tangent is sufficiently close to a cylindrical homotopically non-trivial tangent, then the pseudo-tangent is also cylindrical and homotopically non-trivial. This requires making a pointwise estimate which follows from the derivative estimates given by Schoen-Uhlenbeck [SU82] and Bethuel [Bet93] regularity theorems. This rigidity result relies on the fact that we are comparing a homogeneous degree zero map ϕ to a homotopically non-trivial cylindrical map ψ. The homotopically non-trivial property helps us push singularities from ψ to ϕ, whilst the cylindrical property ensures that there are enough singularities on ψ to push the spine dimension of ϕ up to some maximal value.

The Reifenberg approximation property requires that the approximating planes do not tilt too much as you change base point and scale. As such we need to show that the spines of the pseudo-tangents vary continuously with respect to the base point and scale. We also need this to compare two pseudo-tangents at different base points and scales, in particular in the case that we already know one of these pseudo-tangents is cylindrical and homotopically non-trivial. To apply rigidity we would need that these two pseudo-tangents are close, which can be arranged if the pseudo-tangents are at points and scales that are close to each other, subject to proving some continuity with respect to changing base point and scale. Such a continuity result is shown in section 5.4.

Finally we pull together these results in section 5.5 and prove the structure result for the singular set. By assuming the existence of a cylindrical homotopically non-trivial tangent with minimal cylindrical density at some fixed singularity, we proceed by showing the pseudo-tangents inherit the cylindrical and homotopically non-trivial properties by the rigidity result and an iterative method. However pseudo-tangents were only defined on part of the singular set, so we also prove a no-gaps lemma to show that local to such a singularity, this subset of the singular set is in fact the whole singular set.

The method outlined above extends to the case of mean curvature flow with some suitable modifications. There are however three primary differences, a rigidity type result is only known for a particular type of tangent, in general only a subset of the singular set local to a particular singularity can be shown to have a structure result, and the structure result is only that this subset is contained in a finite union of Lipschitz submanifolds and a lower dimensional set. This argument is discussed in the chapters following and including Chapter 6.

Chapter 2

Background Material

2.1 Notation and Definitions

Integers $n \geq 2$ and $m \geq 2$ will often denote the dimension of the domain and target of maps. For example $u : \Omega \to N$ where $\Omega \subset \mathbb{R}^n$ is an open subset and N is an m-dimensional smooth Riemannian manifold. Often for simplicity we will work with the case that Ω is the unit ball centred on the origin. Further we may assume N is isometrically embedded in Euclidean space by the Nash embedding theorem [Nas56].

We use the following notation to denote neighbourhoods of a sets, including balls.

Definition 2.1.1 (Neighbourhoods and balls)**.** Let $S \subset \mathbb{R}^k$ for an integer $k > 0$, and $r > 0$ a radius. We denote the r-neighbourhoods of S as follows.

$$B_r(S) = \{x \in \mathbb{R}^k \, : \, \mathrm{dist}(x, S) < r\}, \quad \overline{B}_r(S) = \{x \in \mathbb{R}^k \, : \, \mathrm{dist}(x, S) \leq r\}.$$

When $S = \{x\}$ we write $B_r(x)$ for the open ball around x, and when $x = 0$ we write B_r. When it is necessary to distinguish the dimension, for example when $k \neq n$, we may write $B_r^k(x)$ for the k-dimensional ball in $\mathbb{R}^k$ centred on x. For $0 < k \leq n$, and a k-dimensional subspace $L \subset \mathbb{R}^n$ we write $\{x\} \times B_r^{n-k} \subset L \times L^\perp = \mathbb{R}^n$ for the $(n-k)$-dimensional ball of radius r contained in $L^\perp$, centred on $x \in L$.

We denote the k-dimensional sphere of radius r centred on $x \in \mathbb{R}^{k+1}$ by $S_r^k(x) = \partial B_r^{k+1}(x)$. For brevity the k-dimensional sphere of radius r centred at the origin is denoted S_r^k and when $r = 1$ we simply write S^k.

Remark 2.1.2. Since we identify a smooth Riemannian target manifold N with a subset in Euclidean space, we can denote by $B_r(N)$ the radius r-neighbourhood of this embedding of N in Euclidean space.

We will need the following standard measures on Euclidean space.

Definition 2.1.3 (Lebesgue and Hausdorff measures)**.** The Lebesgue measure on $\mathbb{R}^n$ is denoted $\mathcal{L}^n$. The k-dimensional Hausdorff measures on $\mathbb{R}^n$ are denoted by $\mathcal{H}^k$. The volume of k-dimensional unit ball is defined as $\omega_k = \mathcal{L}^k(B_1^k)$. The Hausdorff dimension of a subset $A \subset \mathbb{R}^n$ is denoted by $\dim_{\mathcal{H}}(A) = \inf\{k : \mathcal{H}^k(A) = 0\}$ for a subset $A \subset \mathbb{R}^n$.

Definition 2.1.4 (Grassmanian)**.** The collection of k-dimensional subspaces of $\mathbb{R}^n$ is denoted $G_k(n)$. Given $L_i \in G_k(n)$ we say L_i converge to $L \in G_k(n)$ if the basis vectors of L_i converge to the basis vectors of L after suitable rearrangement.

Remark 2.1.5. Of course $G_k(n)$ is compact under this convergence, as the collection of unit norm vectors is compact.

Definition 2.1.6 (Compactly Contained)**.** Given a set $A \subset B$ we say $A \subset\subset B$ if $\bar{A}$ is compact and $\bar{A} \subset \mathring{B}$, where $\bar{A}$ denotes the closure of A, and $\mathring{B}$ the interior of B. In the case of bounded open sets $\widetilde{\Omega} \subset\subset \Omega$ this is equivalent to

$$\inf\left\{|x-y| \,:\, x \in \widetilde{\Omega}, y \in \partial\Omega\right\} > 0.$$

Definition 2.1.7 (Hausdorff Distance)**.** Given two sets $X, Y \subset \mathbb{R}^n$ the Hausdorff distance between X and Y is defined by

$$\operatorname{dist}_{\mathcal{H}}(X,Y) = \inf\{\epsilon > 0 \,:\, X \subset \bar{B}_\epsilon(Y) \text{ and } Y \subset \bar{B}_\epsilon(X)\}.$$

Remark 2.1.8. Subspaces $L_i \in G_k(n)$ converge to L as in Definition 2.1.4 if and only if L_i converge to L in Hausdorff distance.

The following simple geometric proposition will be useful later.

Proposition 2.1.9. *[Nearby Subspaces] For any $\epsilon > 0$ and $k = 1, 2, \ldots, n-1$ there is $\delta = \delta(n,k,\epsilon) > 0$ such that the following holds. Suppose $L, M \in G_k(n)$ and*

$$L \cap B_1 \subset B_\delta(M) \cap B_1.$$

Then

$$L^\perp \cap B_{1/2} \subset B_\epsilon(M^\perp) \cap B_{1/2}.$$

Remark 2.1.10. In particular the unit sphere on $L^\perp$ is a small rotation of the unit sphere on $M^\perp$.

Proof. If this were not the case we could take sequences $L_i, M_i \in G_k(n)$ such that both L_i and M_i converge to $L \in G_k(n)$. Further we could take $x_i \in L_i^{\perp} \cap B_{1/2}$ with $\operatorname{dist}(x_i, M_i^{\perp}) \geq \epsilon$ for some fixed $\epsilon > 0$. Clearly we can take a convergent subsequence so that $x_i \to x \in L^{\perp} \cap B_1$ contrary to the fact that

$$\operatorname{dist}(x, L^{\perp}) \geq \liminf_{i\to\infty} \operatorname{dist}(x_i, M_i^{\perp}) \geq \epsilon.$$

□

Definition 2.1.11 (C^p-spaces)**.** Given an open set $\Omega \subset \mathbb{R}^n$, and $N \subset \mathbb{R}^k$ we denote by $C^p(\Omega; N)$ the set of functions with p-continuous derivatives, and values in N. Given $u \in C^p(\Omega; \mathbb{R}^k)$ denote the support of u by

$$\operatorname{spt}(u) = \{x \in \Omega \,:\, u(x) \neq 0\}.$$

We denote by $C_c^p(\Omega; \mathbb{R}^k)$ the set of $u \in C^p(\Omega; \mathbb{R}^k)$ with $\operatorname{spt}(u) \subset\subset \Omega$.

Definition 2.1.12 ($C^{p,\alpha}$-spaces)**.** Given an open subset $\Omega \subset \mathbb{R}^n$, $N \subset \mathbb{R}^k$ and $\alpha \in (0,1)$ we denote by $C^{p,\alpha}(\Omega; N)$ the set of $u \in C^p(\Omega; N)$ such that the Hölder norm with exponent α of the first p derivatives of u is bounded. In particular when $p = 0$ this is the collection of continuous N-valued functions on Ω such that

$$\sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} < \infty.$$

Definition 2.1.13 ($W^{1,2}$-spaces)**.** Given an open subset $\Omega \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ we denote by $W^{1,2}(\Omega; N)$ the set of L^2-bounded functions with L^2-bounded weak derivatives, and with $u(x) \in N$ for $\mathcal{L}^n$-almost every $x \in \Omega$. We say $u \in W^{1,2}_{loc}(\Omega; N)$ if $u \in W^{1,2}(K; N)$ for any $K \subset\subset \Omega$. We denote by $W_0^{1,2}(\Omega; \mathbb{R}^k)$ the closure of $C_c^{\infty}(\Omega; \mathbb{R}^k)$ in the $W^{1,2}$-norm. In particular given a bounded open set $\Omega \subset \mathbb{R}^n$ and functions $u, v \in W^{1,2}(\Omega; N)$ we say $u = v$ on $\partial\Omega$ if $u - v \in W_0^{1,2}(\Omega; \mathbb{R}^k)$.

Definition 2.1.14 (C^0 and $W^{1,2}$ norms)**.** Given $u : \Omega \to N$ we define the following notations for norms.

$$\begin{aligned}
\|u\|_{W^{1,2}(\Omega)} &= \left(\int_{\Omega} |u|^2 + |Du|^2 \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2}}, \quad u \in W^{1,2}(\Omega; N), \\
\|u\|_{C^0(S)} &= \sup_{x\in S} |u(x)|, \quad S \subset \Omega \text{ is a closed set, } u \in C^0(S).
\end{aligned}$$

The $W^{1,2}$ Sobolev spaces satisfy an important compactness property.

Theorem 2.1.15 (Rellich-Kondrachov Compactness)**.** *Given a bounded open subset $\Omega \subset \mathbb{R}^n$, and a sequence $u_i \in W^{1,2}(\Omega; N)$ with $\sup_i \|u_i\|_{W^{1,2}(\Omega)} < \infty$, there is a subsequence of u_i that converge strongly in L^2 and weakly in $W^{1,2}$ to a map $u \in W^{1,2}(\Omega; N)$.*

We will also need to make use of Radon measures on subsets of $\mathbb{R}^n$.

Definition 2.1.16 (Radon Measures)**.** A measure μ on a set $\Omega \subset \mathbb{R}^n$ is Radon if $\mu(K) < \infty$ for any $K \subset\subset \Omega$. If a statement holds for all points in $\Omega \setminus Z$ where $\mu(Z) = 0$ we say this statement holds on Ω almost everywhere with respect to μ, often written in the shorthand μ-a.e.

Given a Radon measure μ on Ω and a test function $f \in C_c^0(\Omega)$ we use the following notation

$$\mu(f) = \int_\Omega f \, \mathrm{d}\mu.$$

A sequence μ_i of Radon measures on Ω is said to converge weakly to a Radon measure μ on Ω if $\mu_i(f) \to \mu(f)$ for all $f \in C_c^0(\Omega)$.

The following is a standard and useful fact about convergence of Radon measures.

Proposition 2.1.17 (Convergence of Radon Measures)**.** *Suppose μ_i are Radon measures on $\Omega \subset \mathbb{R}^n$ and $\mu_i \to \mu$ another Radon measure on Ω. Then for any $x \in \Omega$ and almost every $r \in (0, \operatorname{dist}(x, \partial\Omega))$ we have that $\mu_i(B_r(x)) \to \mu(B_r(x))$.*

Proof. To prove this let $\phi_j, \psi_j \in C_c^\infty(\Omega)$ be test functions such that ϕ_j is equal to 1 on $B_{r-1/j}(x)$ and equal to 0 outside $B_r(x)$, and ψ_j is equal to 1 on $\overline{B}_r(x)$ and 0 outside $B_{r+1/j}(x)$. Further suppose ϕ_j and ψ_j only take values between 0 and 1. Then $\psi_j - \phi_j \to \chi_{\partial B_r(x)}$ converge pointwise, where $\chi_{\partial B_r(x)}$ is the characteristic function of $\partial B_r(x)$. Then by dominated convergence theorem we have that $\mu(\psi_j - \phi_j) \to \mu(\partial B_r(x))$. We also have that

$$\mu_i(\phi_j) \to \mu(\phi_j) \leq \mu(B_r(x)).$$

Also

$$\mu_i(\psi_j) \to \mu(\psi_j) \geq \mu(B_r(x)).$$

Then since $\mu_i(\phi_j) \leq \mu_i(B_r(x)) \leq \mu_i(\psi_j)$ we have that

$$\limsup_{i\to\infty} \mu_i(B_r(x)) - \liminf_{i\to\infty} \mu_i(B_r(x)) \leq \limsup_{j\to\infty} \mu(\psi_j - \phi_j) = \mu(\partial B_r(x)). \tag{2.1}$$

Now by considering the set $S_{k,R} = \{r \in (0, R) : \mu(\partial B_r(x)) \geq 1/k\}$ we see that since μ is Radon, $S_{k,R}$ is finite for each fixed $k > 0, R > 0$. As such the set

$$S = \bigcup_{k=1}^{\infty} \bigcup_{R=1}^{\infty} S_{k,R}$$

is countable, and for each $r \in (0, \mathrm{dist}(x, \partial\Omega)) \setminus S$ we have that $\mu(\partial B_r(x)) = 0$. This implies $\mu_i(B_r(x)) \to \mu(B_r(x))$ by (2.1). □

Another way to prove this is to note that $\mu(B_r(x))$ is monotonic in r, and so is continuous almost everywhere. We can then approximate by test functions in a similar manner.

The weak convergence of Radon measures is compact in the following sense, due to the dual representation of Radon measures as linear functionals on non-negative compactly supported test functions.

Theorem 2.1.18 (Compactness of Radon measures)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset, and μ_i Radon measures on Ω satisfying $\sup_i \mu_i(K) < \infty$ for each $K \subset\subset \Omega$. Then there is a Radon measure μ on Ω and a subsequence of μ_i such that $\mu_i \to \mu$ converge weakly.*

Given a map $u \in W^{1,2}_{loc}(\Omega; N)$ we may define a Radon measure, called the energy measure, of u.

Definition 2.1.19 (Energy Measure)**.** Let $u \in W^{1,2}_{loc}(\Omega; N)$. The energy measure of u is $\mu = |Du|^2 \mathrm{d}x$. This can be defined by

$$\mu(\phi) = \int_\Omega \phi |Du|^2 \, \mathrm{d}x, \quad \text{for } \phi \in C^0_c(\Omega).$$

Remark 2.1.20. Clearly $|Du|^2\mathrm{d}x$ is a Radon measure as $|Du|$ is locally L^2 on Ω.

The following is Fatou's lemma, a useful result of measure theory that we will make use of occasionally.

Lemma 2.1.21 (Fatou's Lemma)**.** *Let (Ω, Σ, μ) denote a measure space, and $f_i : \Omega \to [0, \infty]$. Then*

$$\int_\Omega \liminf_{i\to\infty} f_i \, \mathrm{d}\mu \leq \liminf_{i\to\infty} \int_\Omega f_i \, \mathrm{d}\mu.$$

Remark 2.1.22. Often we will know $\liminf_{i\to\infty} f_i$, for example if f_i converge pointwise to some function almost everywhere.

We will need to metrise weak convergence of locally bounded Radon measures on a domain Ω. In fact this follows from Banach-Alaoglu compactness, since Radon measures are the dual space of positive linear functionals on $C^0_c(\Omega)$.

Proposition 2.1.23 (Metrisation of weak convergence of Radon measures)**.** *Let $\Lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set. Consider the collection of Radon measures μ on Ω such that $\mu(K) \leq \Lambda$ for any $K \subset\subset \Omega$. The weak convergence of these Radon measures is metrisable.*

Proof. Let $\phi_i \in C^0_c(\Omega)$ be bounded by 1, that is $|\phi_i| \leq 1$, and suppose ϕ_i are dense in the collection of all $\phi \in C^0_c(\Omega)$ with $|\phi| \leq 1$. Then given Radon measures μ, ν on Ω, bounded locally by Λ, we define

$$d(\mu, \nu) = \sum_{i=1}^{\infty} 2^{-i} |\mu(\phi_i) - \nu(\phi_i)|.$$

Then if $\mu_j \to \mu$ we have that $\mu_j(\phi_i) \to \mu(\phi_i)$ for each fixed i as $j \to \infty$. Further for any $\epsilon > 0$ there is $I = I(\epsilon, \Lambda) > 0$ such that

$$\sum_{i=I}^{\infty} 2^{-i} |\mu_j(\phi_i) - \mu(\phi_i)| \leq \sum_{i=I}^{\infty} 2^{1-i} \Lambda < \epsilon.$$

As such we have that $d(\mu_j, \mu) \to 0$ when $\mu_j \to \mu$. Conversely suppose $d(\mu_j, \mu) \to 0$ and suppose $\phi \in C^0_c(\Omega)$. Since ϕ is compactly supported, $\hat{\phi} = \phi / |\phi|_{C^0(\Omega)}$ is well defined and satisfies $|\hat{\phi}| \leq 1$. Therefore a subsequence of ϕ_i converge to $\hat{\phi}$ uniformly, and so we have

$$|\mu_j(\phi) - \mu(\phi)| \leq |\phi|_{C^0(\Omega)} \left(2\Lambda |\hat{\phi} - \phi_i| + 2^i d(\mu_j, \mu) \right).$$

This is null as ϕ_i converge uniformly to ϕ and $d(\mu_j, \mu) \to 0$. As such $\mu_j \to \mu$ as $\phi \in C^0_c(\Omega)$ we arbitrary. □

Definition 2.1.24 (Metrisation of weak convergence of Radon measures)**.** Given $\Lambda > 0$ and an open subset $\Omega \subset \mathbb{R}^n$ let d denote a metrisation of the weak converge of Radon measures μ on Ω with $\mu(K) \leq \Lambda$ for each $K \subset\subset \Omega$.

Remark 2.1.25. Note that the set Ω and bound Λ is not explicit in this notation, however in practice it will always be clear from the context.

Recall that we say $f_i \in W^{1,2}(\Omega; \mathbb{R}^k)$ converge weakly to $f \in W^{1,2}(\Omega; \mathbb{R}^k)$ if the following holds.

$$\int_\Omega Df_i \phi \to \int_\Omega Df \phi, \quad \text{for all } \phi \in C^1_c(\Omega).$$

We may metrize this weak convergence in $W^{1,2}(\Omega;\mathbb{R}^k)$ in a similar manner. Again this follows from Banach-Alaoglu since $W^{1,2}(\Omega;\mathbb{R}^k)$ is its own dual space.

Proposition 2.1.26 (Metrisation of weak convergence in $W^{1,2}$)**.** *Let $\Lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be an open subset. There is a metrisation of the weak convergence of $f_i \in W^{1,2}_{loc}(\Omega;\mathbb{R}^k)$ with $|Df_i|_{L^2(K)} \leq \Lambda$ for each $K \subset\subset \Omega$.*

Proof. Again let $\phi_i \in C^1_c(\Omega)$ be bounded $|\phi_i| \leq 1$ and dense in the set of all $\phi \in C^1_c(\Omega)$ with $|\phi| \leq 1$. Then we can define

$$d_{W^{1,2}}(f,g) = \sum_{i=1}^{\infty} 2^{-i} \left| \int_\Omega Df\phi_i \,\mathrm{d}x - \int_\Omega Dg\phi_i \,\mathrm{d}x \right|.$$

If $d_{W^{1,2}}(f_j,f) \to 0$ then for any $\phi \in C^1_c(\Omega)$, set $\hat{\phi} = \phi/|\phi|_{C^0(\Omega)}$. We can find a subsequence of ϕ_i such that $\phi_i \to \hat{\phi}$ uniformly. Then we have that

$$\left| \int_\Omega Df_j\phi \,\mathrm{d}x - \int_\Omega Df\phi \,\mathrm{d}x \right| \leq |\phi|_{C^0_c(\Omega)} \left(2\Lambda\mathcal{L}^n(\Omega)^{\frac{1}{2}} |\hat{\phi} - \phi_i|_{C^0(\Omega)} + 2^i d(f_j,f) \right).$$

This is null as $\phi_i \to \hat{\phi}$ converges uniformly, and $d_{W^{1,2}}(f_j,f) \to 0$. Now conversely suppose f_j converge weakly to f in $W^{1,2}$. For any $\epsilon > 0$ there is $I = I(\Lambda,\epsilon) > 0$ such that

$$\sum_{i=I}^{\infty} 2^{-i} \left| \int_\Omega Df_j\phi_i \,\mathrm{d}x - \int_\Omega Df\phi_i \,\mathrm{d}x \right| \leq \epsilon.$$

As such it follows that $d_{W^{1,2}}(f_j,f) \to 0$, as each of the first I terms in the sum converge to 0, and the remaining part of the series is null by the above. □

Definition 2.1.27 (Metrisation of weak convergence in $W^{1,2}$)**.** Let $\Omega \subset \mathbb{R}^n$ be an open subset. We denote by $d_{W^{1,2}}$ a metrisation of the weak convergence of functions in $W^{1,2}(\Omega;\mathbb{R}^k)$.

The following basic topology will be useful later. Recall $S^k \subset \mathbb{R}^{k+1}$ is the k-dimensional sphere.

Definition 2.1.28 (Homotopy classes)**.** Let $f,g \in C^0(S^k;N)$ be continuous maps. We say f and g are homotopically equivalent if there exists a map $F \in C^0(S^k \times [0,1];N)$ such that

$$F_0(\cdot) = F(\cdot,0) = f, \quad F_1(\cdot) = F(\cdot,1) = g.$$

Such a map F is called a homotopy. Homotopy equivalence is an equivalence relation. We will say $f \in C^0(S^k;N)$ is homotopically trivial if it is homotopically equivalent to a constant map. We will say f is homotopically non-trivial if it is not homotopically equivalent to a constant.

Remark 2.1.29. Of course we can replace the unit sphere S^k in this definition by any radius sphere S^k_r. Later for a function f defined on a ball B_r we will consider whether the restriction of f to S^{n-1}_ρ is homotopically trivial for $0 < \rho \leq r$.

Note that the homotopy must take values in N also, any map from S^k to Euclidean space could easily be contracted to a point by homotopy without this restriction.

The homotopy classes can be shown to be rigid under pointwise closeness. To prove this we make use of the existence of a smooth nearest point projection onto a manifold. This result is proved in appendix 2.12.3 of Simon's book [Sim96b].

Proposition 2.1.30 (Nearest Point Projection)**.** *Let N be a smooth compact manifold of dimension m embedded in $\mathbb{R}^k$. Then there is $\tau = \tau(N) > 0$ and a smooth projection $\Pi : B_\tau(N) \to N$ such that $|\Pi(y) - y| = \operatorname{dist}(y, N)$ and $|z - y| > \operatorname{dist}(y, N)$ for any $z \in N \setminus \Pi(y)$, and $y \in B_\tau(N)$.*

Remark 2.1.31. In the case that N is analytic, the projection can also be made analytic. In the case that N is C^p for $p < \infty$, then the projection can be made to be C^{p-1}.

Using this we can prove the following rigidity for homotopy classes.

Proposition 2.1.32 (Homotopy Equivalence of Pointwise Close Maps)**.** *There is $\tau = \tau(N) > 0$ such that if $f, g \in C^0(S^k; N)$ and $\sup_{x \in S^k} |f(x) - g(x)| < \tau$ then f is homotopically equivalent to g.*

Proof. Choose $\tau > 0$ sufficiently small that Proposition 2.1.30 applies. Then $F_s = \Pi((1 - s)f + sg)$ is a well defined homotopy from f to g. $\square$

This allows us to push singularities from homotopically non-trivial maps to other maps that are pointwise close. First observe that if $f : B_2 \to N$ is continuous then we can define a homotopy between f and $f(0)$ simply by $F_r(x) = f(rx)$ for $r \in [0, 1]$. As such f is homotopically trivial on any sphere S^{n-1}_ρ for $\rho \in [0, 2)$. So any map $g : B_2 \to N$ that is homotopically non-trivial on S^{n-1}_ρ for some $\rho \in [0, 2)$ must be discontinuous at some point in B_ρ. Now suppose $f, g : B_2 \to N$, with g homotopically non-trivial on S^{n-1}_ρ for some $\rho \in [0, 2)$. If f is sufficiently pointwise close to g on S^{n-1}_ρ then f is also homotopically non-trivial, and f is discontinuous at some point in B_ρ.

Later we apply this idea to slices of stationary harmonic maps along some linear subspace.

Another useful result is that a homotopy on the domain passes to the maps, if the maps are regular at all points of the domain homotopy. Later this will help us show that if a map is homotopically non-trivial when restricted to some sphere, then it is homotopically non-trivial when restricted to a nearby sphere.

Proposition 2.1.33 (Domain Homotopies)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is an open subset and $S_1, S_2 \subset \Omega$ are two k-dimensional spheres, with $k < n$. Given a map $f : \Omega \to N$, suppose f is continuous on some subset $\widetilde{\Omega} \subset \Omega$. Further suppose there is a homotopy $F : S^k \times [0,1] \to \Omega$ such that $F_0(S^k) = S_1$, $F_1(S^k) = S_2$, and $F_t(x) \in \widetilde{\Omega}$ for all $t \in [0,1]$ and $x \in S^k$. Then $f \lfloor S_1$ is homotopically equivalent to $f \lfloor S_2$.*

Remark 2.1.34. Later we will use this when f is the limit map of a sequence of stationary harmonic maps f_i, and $\widetilde{\Omega}$ is a subset of the regular set of f.

Proof. This follows since $G_t(x) = f(F_t(x))$ is a homotopy between $f \lfloor S_1$ and $f \lfloor S_2$. The main point is that $F_t(x)$ only passes through points where f is known to be smooth. $\square$

2.2 The Reifenberg Topological Disk Theorem

The Reifenberg theorem states that a closed set can be mapped to a disk by a bi-Hölder continuous homeomorphism, given the set satisfies an appproximation property by planes. To be clear we define a bi-Hölder homeomorphism as follows.

Definition 2.2.1 (Bi-Hölder Homeomorphism)**.** A map $f : X \to Y$ between two metric spaces X, Y is called a bi-Hölder homeomorphism with exponent $\beta \in (0,1)$ if f is a homeomorphism, $f \in C^{0,\beta}(X;Y)$ and $f^{-1} \in C^{0,\beta}(Y,X)$.

The Reifenberg theorem was first proved by Reifenberg [Rei60]. The statement and a proof can also be found in Simon's notes on the Reifenberg theorem [Sim96a]. First we define the ϵ-Reifenberg approximation property. Recall $\mathrm{dist}_{\mathcal{H}}$ denotes the Hausdorff distance between sets.

Definition 2.2.2 (Approximation Property)**.** Let $\epsilon > 0$ and $S \subset \overline{B}_2$ a closed subset containing 0. We say S satisfies the m-dimensional ϵ-Reifenberg approximation condition in $\overline{B}_1$ if for each $x \in S \cap \overline{B}_1$ and $r \in (0,1]$ there is $L_{x,r} \in G_m(n)$ such that

$$\mathrm{dist}_{\mathcal{H}}(S \cap \overline{B}_r(x), x + L_{x,r}) < \epsilon.$$

Theorem 2.2.3 (Reifenberg's Theorem)**.** *There is $\epsilon = \epsilon(n) > 0$ such that for any closed set $S \subset \overline{B}_2$ containing 0 satisfying the m-dimensional ϵ-Reifenberg approximation condition in $\bar{B}_1$, $S \cap \overline{B}_1$ is homeomorphic to $\overline{B}_1^m$.*

Further there is a closed set $M \subset \mathbb{R}^n$ such that $M \cap \overline{B}_1 = S \cap \overline{B}_1$ and a homeomorphism $\tau : T \to M$ for some $T \in G_m(n)$, with the following properties. There is $C = C(n) > 0$ such that

$$\sup_{x,y \in T, x \neq y} \frac{|\tau(x) - \tau(y)|}{|x - y|} \leq C(n)\epsilon,$$

and $\tau \lfloor T \setminus \overline{B}_2 = \mathrm{Id}$.

Given any $\beta \in (0,1)$ we can further find $\epsilon = \epsilon(n, \beta) > 0$ such that the above holds, and τ is a bi-Hölder homeomorphism with exponent β.

If one makes the weaker assumption that the closed set S is only locally contained in neighbourhoods of planes, and these planes do not tilt too much as you change base point and scale, then it can be shown that S is contained in the bi-Hölder image of a disk. Such a result was proved by David-Toro [DT12]. We will state this here. The statement makes use of the following normalized Hausdorff distance.

Definition 2.2.4 (Normalised Hausdorff distance)**.** Let $E, F \subset \mathbb{R}^n$ be closed subsets. For any $r > 0$ and $x \in \mathbb{R}^n$ such that $B_r(x)$ meets E and F, we define

$$\mathrm{dist}_{x,r}(E, F) = \frac{1}{r} \max \left\{ \sup_{y \in E \cap B_r(x)} \mathrm{dist}(y, F), \sup_{y \in F \cap B_r(x)} \mathrm{dist}(y, E) \right\}.$$

In the follow $\mathbb{N}_0$ denotes the non-negative integers.

Theorem 2.2.5 (David-Toro Reifenberg Theorem)**.** *For any integer $d \in \mathbb{N}_0$ and $\tau \in (0, 1/10)$ there is $\epsilon > 0$ such that the following holds. Suppose $E \subset B_1$ is a closed subset with $0 \in E$. For each $y \in E$ and $s \in [0, 10]$ suppose there is a d-dimensional plane $L_{y,s}$ through y such that*

$$E \cap B_s(x) \subset B_{\epsilon s}(L_{y,s}).$$

Further suppose we have the following tilt conditions.

$$\mathrm{dist}_{y,10^{-k}}(L_{x,10^{-k}}, L_{x,10^{-k+1}}) \leq \epsilon, \quad y \in E, k \in \mathbb{N}_0.$$

$$\mathrm{dist}_{y,10^{-k}}(L_{y,10^{-k}}, L_{x,10^{-k}}) \leq \epsilon, \quad x, y \in E, |x - y| \leq 10^{-k+2}, k \in \mathbb{N}_0.$$

Then there is a bijective map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $|g(x) - x| \leq \tau$ for all $x \in \mathbb{R}^n$, and

$$\frac{1}{4}|x - y|^{1+\tau} \leq |g(x) - g(y)| \leq 3|x - y|^{1-\tau},$$

such that $E \subset g(L_{0,10})$.

Chapter 3

Energy Minimising and Stationary Harmonic Maps

3.1 Definitions

We first consider energy minimising maps. Throughout this section $\Omega \subset \mathbb{R}^n$ will denote an open subset and N an m-dimensional manifold embedded in $\mathbb{R}^k$ for some $k \geq m$.

Definition 3.1.1 (Energy Minimising Map)**.** The energy of $u \in W^{1,2}_{loc}(\Omega; N)$ on $\widetilde{\Omega} \subset\subset \Omega$ is defined by

$$\mathcal{E}_{\widetilde{\Omega}}(u) = \|Du\|_{L^2(\widetilde{\Omega})} = \int_{\widetilde{\Omega}} |Du|^2 \,\mathrm{d}\mathcal{L}^n.$$

We say u is energy minimising in Ω if for every $B = B_r(x) \subset\subset \Omega$ we have that $\mathcal{E}_B(u) \leq \mathcal{E}_B(w)$ for any $w \in W^{1,2}(B; N)$ with $u = w$ on ∂B.

By computing the first variation of the energy functional $\mathcal{E}_B$ we can find elliptic partial differential equations satisfied by energy minimising maps. For an energy minimising map $u \in W^{1,2}_{loc}(\Omega; N)$ one can make both domain variations and target variations.

Definition 3.1.2 (Domain and Target variations)**.** A domain variation is a variation of the form

$$u_s(y) = u(y + s\zeta(y)), \quad \text{for } \zeta \in C^\infty_c(B_r(x); \mathbb{R}^n).$$

Here we can take $s \in (-\epsilon, \epsilon)$ with $\epsilon > 0$ sufficiently small depending on ζ so that $y + s\zeta(y) \in B_r(x)$ for each $y \in B_r(x)$. This is possible since ζ is compactly supported in $B_r(x)$.

Recall from Proposition 2.1.30 that we can define a smooth projection Π_N onto N from a neighbourhood of N. A target variation is a variation of the form

$$u_s(y) = \Pi_N(u(y) + s\widetilde{\zeta}(y)), \quad \text{for } \widetilde{\zeta} \in C_c^\infty(B_r(x); \mathbb{R}^p).$$

This is well defined for $s \in (-\epsilon, \epsilon)$ for sufficiently small $\epsilon > 0$ depending on ζ by Proposition 2.1.30.

Computing the first variation with respect to these variations gives the following weak differential equations which are satisfied by an energy minimiser. In the following δ_{ij} denotes the Kronecker delta, that is $\delta_{ij} = 0$ whenever $i \neq j$ and $\delta_{ii} = 1$.

Proposition 3.1.3 (Variational Equations)**.** *Suppose $u \in W^{1,2}_{loc}(\Omega; N)$ satisfies*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{E}_B(u_s) = 0, \quad \textit{for } B \subset\subset \Omega, \tag{3.1}$$

where u_s is a target variation. Then u satisfies the following weak differential equation.

$$\int_\Omega \sum_{i=1}^n D_i u \cdot D_i \zeta - \zeta \cdot A_u(D_i u, D_i u) \, \mathrm{d}\mathcal{L}^n = 0, \quad \zeta \in C_c^\infty(\Omega; \mathbb{R}^p). \tag{3.2}$$

Now suppose (3.1) *holds for a domain variation u_s. Then u satisfies the following weak differential equation.*

$$\int_\Omega \sum_{i=1}^n \left(|Du|^2 \delta_{ij} - 2 D_i u \cdot D_j u\right) D_i \zeta_j \, \mathrm{d}\mathcal{L}^n = 0, \quad \zeta \in C_c^\infty(\Omega; \mathbb{R}^p). \tag{3.3}$$

Weak solutions to (3.2) and (3.3) define weakly harmonic and stationary harmonic maps.

Definition 3.1.4 (Weakly and Stationary Harmonic Maps)**.** If $u \in W^{1,2}_{loc}(\Omega; N)$ satisfies (3.2) we say u is weakly harmonic.

If u also satisfies (3.3) we say u is stationary harmonic. Given $\Lambda > 0$ we denote by $H_\Lambda(\Omega; N)$ the collection of all stationary harmonic maps such that $\mathcal{E}_\Omega(u) \leq \Lambda$.

Remark 3.1.5. Note that a stationary harmonic map doesn't even necessarily have bounded energy on Ω as the derivative is only assumed to be locally L^2 bounded. Whilst this is sufficient to define stationary harmonic maps, it is often useful to have the uniform energy bound given by $H_\Lambda(\Omega; N)$.

In fact it can be shown that stationary harmonic maps are precisely those maps $u \in W^{1,2}_{loc}(\Omega; N)$ for which the energy functional is stationary with respect to an arbitrary variation.

Definition 3.1.6 (Variation)**.** Given $u \in W^{1,2}_{loc}(\Omega; N)$ and a ball $B_r(x) \subset\subset \Omega$, a variation of u on $B_r(x)$ is a one-parameter family $\{u_s\}_{s\in(-\epsilon,\epsilon)}$ for any $\epsilon > 0$ satisfying the following. The maps $u_s \in W^{1,2}(B_r(x); N)$, the family of maps is C^1 in s, and the variation satisfies $u_0 = u$, $u_s(y) = u(y)$ for all $y \in B_r(x) \setminus K$ for some $K \subset\subset B_r(x)$.

Remark 3.1.7. The condition that $u_s(y) = u(y)$ away from a compactly contained set $K \subset\subset B_r(x)$ is to say that the variation is compactly supported on $B_r(x)$. It allows us to avoid variations at the boundary $\partial B_r(x)$.

Proposition 3.1.8 (Equivalent definition of stationary harmonic map)**.** *A map $u \in W^{1,2}_{loc}(\Omega; N)$ is stationary harmonic if and only if*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{E}(u_s) = 0$$

for any compactly supported variation u_s on any ball $B \subset\subset \Omega$.

Stationary harmonic maps may have singularities. We define the following regular and singular sets.

Definition 3.1.9 (Regular and Singular Sets)**.** Let $u \in W^{1,2}_{loc}(\Omega; N)$ be a stationary harmonic map. The regular set $\mathrm{Reg}(u)$ is defined as the set of points $x \in \Omega$ such that u is smooth on an open neighbourhood of x. The singular set is the complement of the regular set $\mathrm{Sing}(u) = \Omega \setminus \mathrm{Reg}(u)$.

Remark 3.1.10. Evidently one has that the regular set is open and the singular set is closed in Ω.

Note that since stationary harmonic maps are defined as solutions to the weak differential equation (3.3), the set of stationary harmonic maps is closed under $W^{1,2}$-limits. However in general there is no compactness for this strong $W^{1,2}$ convergence in $H_\Lambda(\Omega; N)$.

Proposition 3.1.11 ($H_\Lambda(\Omega; N)$ is closed)**.** *Let $u_i \in H_\Lambda(\Omega; N)$ for some $\Lambda > 0$, and suppose u_i converge in the $W^{1,2}$ norm to u. Then $u \in H_\Lambda(\Omega; N)$.*

3.2 Monotonicity and Compactness

The key tool to study singularities of stationary harmonic and energy minimising maps is a monotonicity formula for the energy density ratios.

Definition 3.2.1 (Energy Density Ratio)**.** Let $u \in W^{1,2}_{loc}(\Omega; N)$ and suppose $B_r(x) \subset\subset \Omega$. The energy density ratio at x with scale r is defined by

$$\Theta_u(x,r) = r^{2-n} \int_{B_r(x)} |Du|^2 \, \mathrm{d}\mathcal{L}^n = r^{2-n} \mathcal{E}_{B_r(x)}(u).$$

In the following we use the following radial distance and radial derivative

$$\rho_x(y) = |x-y|, \quad \frac{\partial u}{\partial \rho_x} = \rho_x^{-1}(x-y) \cdot Du.$$

The following lemma is the monotonicity formula. A proof can be found in Simon's book [Sim96b, 2.4] for energy minimising maps, or Lin's paper for stationary harmonic maps [Lin99]. It can be proved by substituting a radial test function $\zeta(y) = x - y$ into (3.3).

Lemma 3.2.2 (Monotonicity Formula)**.** *Let $u \in W^{1,2}_{loc}(\Omega; N)$ be a stationary harmonic map. For any $x \in \Omega$ and $0 < s \leq r < dist(x, \partial\Omega)$ we have*

$$\Theta_u(x,r) = \Theta_u(x,s) + 2 \int_{B_r(x) \setminus B_s(x)} \rho_x^{2-n} \left| \frac{\partial u}{\partial \rho_x} \right|^2 \mathrm{d}\mathcal{L}^n. \tag{3.4}$$

Remark 3.2.3. As such $\Theta_u(x,s)$ is non-decreasing for $0 < s < \mathrm{dist}(x, \partial\Omega)$. Note that (3.3) is used to prove the monotonicity, so the distinction between weakly harmonic and stationary harmonic is important here.

Proof. We wish to substitute $\zeta(x) = x - y$ as a test function into (3.3). However this test function does not have compact support, so we need the following fact from the divergence theorem. Fix a vector $v \in \mathbb{R}^n$ and suppose

$$\int_{B_{r_0}(y)} v \cdot D\zeta \, \mathrm{d}x = 0, \quad \text{for all } \zeta \in C_c^\infty(B_{r_0}(y)).$$

Let $r \in (0, r_0)$ and let $\nu(x) = (x-y)/r$. In particular ν is the outward normal to $\partial B_r(y)$. By approximating the characteristic function on $B_r(y)$ we have that

$$\int_{B_r(y)} v \cdot D\zeta = \int_{\partial B_r(y)} \nu \cdot v \zeta \, \mathrm{d}x, \quad \text{for any } \zeta \in C^\infty(\overline{B}_r(y)). \tag{3.5}$$

Now let $a_{ij} = |Du|^2 \delta_{ij} - 2 D_i u \cdot D_j u$. Substituting (3.5) into (3.3) gives the following.

$$\int_{B_r(y)} \sum_{i,j=1}^{n} a_{ij} D_i \zeta_j \,\mathrm{d}x = \int_{\partial B_r(y)} \sum_{i,j=1}^{n} a_{ij} \nu_i \zeta_j \,\mathrm{d}x.$$

Setting $\zeta(x) = x - y$ we have that $D_i\zeta_j = \delta_{ij}$ and so we get the following.

$$(n-2)\int_{B_r(y)} |Du|^2 \,\mathrm{d}x = r^{-1}\int_{\partial B_r(y)} |Du|^2 - 2\left|\frac{\partial u}{\partial \rho_y}\right|^2 \mathrm{d}x.$$

Multiplying this by r^{1-n} we obtain that for any fixed $0 < \tau < r$ the following holds.

$$\frac{\mathrm{d}}{\mathrm{d}r}\Theta_u(y,r) = 2\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{B_r(y)\setminus B_\tau(y)} \rho_y^{2-n}\left|\frac{\partial u}{\partial \rho_y}\right|^2 \mathrm{d}x\right).$$

Integrating this between $0 < s \leq r < \mathrm{dist}(y, \partial\Omega)$ gives (3.4). □

By the remark we know that the limit $\lim_{r\to 0}\Theta_u(x,r)$ exists. This limit is called the energy density at x.

Definition 3.2.4 (Energy Density)**.** Let $u \in W^{1,2}_{loc}(\Omega, N)$ be a stationary harmonic map. We define the energy density at $x \in \Omega$ by

$$\Theta_u(x) = \lim_{r\to 0}\Theta_u(x,r).$$

Remark 3.2.5. Later we will need to extend this energy density to weak limits of energy measures $\mu_i = |Du_i|^2\mathrm{d}x$ associated to a sequence of stationary harmonic maps $u_i \in H_\Lambda(\Omega; N)$. Note that for $\mu = |Du|^2\mathrm{d}x$ we have that

$$\Theta_u(x) = \lim_{r\to 0}\frac{\mu(B_r(x))}{\omega_n r^n}.$$

In such a case we will define Θ_μ so that it is equal to Θ_u.

The density function is upper-semicontinuous in all its arguments as follows. Recall that by Proposition 3.1.11 we have that $W^{1,2}$ limits of $H_\Lambda(\Omega; N)$ maps are also in $H_\Lambda(\Omega; N)$.

Lemma 3.2.6. *[Upper-Semicontinuity of Density] Let $u_j \in H_\Lambda(\Omega; N)$ be a sequence of stationary harmonic maps converging in $W^{1,2}$ to $u \in H_\Lambda(\Omega; N)$, $x_j \in \Omega$ a sequence of points converging to $x \in \Omega$, and $r_j > 0$ a null sequence. Then $\Theta_u(x) \geq \limsup_{j\to\infty}\Theta_{u_j}(x_j, r_j)$.*

Remark 3.2.7. By monotonicity we also have the frequently used result that $\Theta_u(x) \geq \limsup_{j\to\infty}\Theta_{u_j}(x_j)$. Of course one can take constant sequences of x_j or u_j in the lemma to get upper-semicontinuity in each individual argument.

Proof. Given any $\epsilon > 0$ choose $r = r(\epsilon) > 0$ sufficiently small so that

$$\Theta_u(x, 2r) \leq \Theta_u(x) + \epsilon \tag{3.6}$$

Now choose $J = J(\epsilon) > 0$ sufficiently large so that $j \geq J$ implies

$$\|u_j - u\|^2_{W^{1,2}(\Omega)} < \epsilon r^{n-2}, \quad |x_j - x| < \epsilon r, \quad r_j \leq r. \tag{3.7}$$

By monotonicity one has $\Theta_{u_j}(x_j, r_j) \leq \Theta_{u_j}(x_j, r)$. We can translate the base point from x_j to x with the following estimate. Let $\sigma_j = 1 + \frac{|x_j - x|}{r}$

$$\Theta_{u_j}(x_j, r) \leq \sigma_j^{n-2}\Theta_{u_j}(x, r + |x_j - x|) \leq (1+\epsilon)^{n-2}\Theta_{u_j}(x, (1+\epsilon)r). \tag{3.8}$$

Now we use the $W^{1,2}$ bounds from (3.7) to show the following.

$$\begin{aligned}\Theta_{u_j}(x, (1+\epsilon)r) &\leq \Theta_u(x, (1+\epsilon)r) + (1+\epsilon)^{2-n} r^{2-n}\|u_j - u\|^2_{W^{1,2}(\Omega)} \\ &\leq \Theta_u(x, 2r) + (1+\epsilon)^{2-n}\epsilon.\end{aligned}$$

Combining this with (3.8) and (3.6) we have the upper semi-continuity result.

$$\Theta_{u_j}(x_j, r_j) \leq \Theta_u(x, r) + 2\epsilon, \quad \text{for } j \geq J(\epsilon).$$

□

Here we made use of the fact that a sequence $u_j \in H_\Lambda(\Omega; N)$ converging in $W^{1,2}$ will converge to another stationary harmonic map $u \in H_\Lambda(\Omega; N)$, and so u also satisfies the monotonicity. Unfortunately there is not a compactness result for stationary harmonic maps that gives strong $W^{1,2}$ convergence in general. To get a useful compactness result for stationary harmonic maps we must extend $H_\Lambda(\Omega; N)$ to a collection of measures as done by Lin [Lin99].

The class of energy-minimising maps does have a compactness result due to Luckhaus [Luc88], and partial results earlier due to Schoen-Uhlenbeck [SU82] and Hardt-Lin [HL87].

It is illustrative to consider tangent maps for the energy-minimising case, even though we will need to consider a broader class to study tangents of stationary harmonic maps. The existence of tangent maps to energy minimising maps follows from the following compactness result that can be found in Simon's book [Sim96b].

Theorem 3.2.8 (Compactness Theorem for Energy Minimising Maps)**.** *Suppose* $u_i \in W^{1,2}_{loc}(\Omega; N)$ *are energy minimizing with uniformly bounded energy*

$$\sup_{i\geq 1} \mathcal{E}_{B_r(x)}(u_i) < \infty, \quad \textit{for each } B_r(x) \subset\subset \Omega.$$

Then we can find a subsequence of u_i *that converges in* $W^{1,2}$ *to an energy minimizing map* $u \in W^{1,2}_{loc}(\Omega; N)$ *on each ball* $B_r(x) \subset\subset \Omega$.

Remark 3.2.9. The convergence in $W^{1,2}$ on each ball $B \subset\subset \Omega$ is sometimes called local $W^{1,2}$ convergence.

In the case of stationary harmonic maps we of course still have Rellich compactness Theorem 2.1.15. However weak $W^{1,2}$ convergence is not sufficient to imply the limit is stationary harmonic, so in particular the weak $W^{1,2}$ limit may not satisfy monotonicity.

3.3 Tangent Maps and Regularity

Tangent maps are the maps achieved by rescaling an energy minimising map around a point. The tangent maps are particularly useful for studying singularities. We first define how to rescale and translate a map.

Definition 3.3.1 (Rescale and Translation)**.** Given a map $u : \Omega \to N$, $x \in \Omega$ and $\lambda > 0$ we define

$$\Omega_{x,\lambda} = \{\lambda^{-1}(y - x) \,:\, y \in \Omega\} = \lambda^{-1}(\Omega - x).$$

We may then define the rescaled map $u_{x,\lambda} : \Omega_{x,\lambda} \to N$ as

$$u_{x,\lambda}(y) = u(x + \lambda y).$$

Remark 3.3.2. Note that if $B = B_r(x)$ then $B_{x,\lambda} = B_{r/\lambda}(0)$, and as $\lambda \to 0$ this set becomes larger. In other words for any $K \subset\subset \mathbb{R}^n$ and $x \in \Omega$ there is $\delta > 0$ such that for $0 < \lambda < \delta$ we have that $K \subset\subset \Omega_{x,\lambda}$. In this sense $\Omega_{x,\lambda}$ is converging to $\mathbb{R}^n$ as $\lambda \to 0$ for any $x \in \Omega$.

We have the following useful relation between the density of $u_{x,\lambda}$ and u.

Proposition 3.3.3. *Suppose* $u \in W^{1,2}_{loc}(\Omega; N)$ *is stationary harmonic,* $\lambda > 0$ *and* $x \in \Omega$. *Then for any* $r > 0$ *we have that*

$$\Theta_{u_{x,\lambda}}(0, r) = \Theta_u(x, \lambda r).$$

Remark 3.3.4. This computation along with monotonicity is the key to why tangent maps have energy density ratios at the origin that are independent of the scale r.

Tangent maps are a special case of limit maps, the distinction being that limit maps allow the base point of the rescales to vary with the sequence.

Definition 3.3.5 (Limit and Tangent Maps)**.** Given $u \in W^{1,2}_{loc}(\Omega; N)$ and $x \in \Omega$ we say $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ is a limit map of u at x if it can be achieved as the $W^{1,2}$-limit of a sequence u_{x_j,λ_j} with $x_j \in \Omega$, $x_j \to x$ and $\lambda_j > 0$, $\lambda_j \to 0$.

We say ϕ is a tangent map if it can be achieved as the $W^{1,2}$-limit of a sequence u_{x,λ_j} for a null sequence $\lambda_j > 0$. We denote by $T_x u$ the collection of all tangent maps of u at x.

Remark 3.3.6. Note that limit maps are defined on all of $\mathbb{R}^n$, though the $W^{1,2}$-convergence is only local. Also note that for a general $u \in W^{1,2}_{loc}(\Omega; N)$ it is not clear that tangent maps even exist, and when they do if they are unique. The existence is true for energy minimising maps, but the uniqueness is not known in general even in this case.

The existence of limit maps and tangent maps for energy minimising maps follows immediately from compactness Theorem 3.2.8. Note that the uniform energy bound assumption here can be easily verified for a sequence of rescales u_{x_i,λ_i} where $x_i \to x$ and $\lambda_i \to 0$.

Lemma 3.3.7 (Existence of Limits and Tangents)**.** *Let $u \in W^{1,2}_{loc}(\Omega; N)$ be an energy-minimising map. Suppose $x_j \in \Omega$ converge to x, and $\lambda_j > 0$ converge to 0. Then there is a subsequence such that u_{x_j,λ_j} converge to an energy minimising map $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$.*

Remark 3.3.8. In particular $T_x u \neq \emptyset$ for each $x \in \Omega$ where $u \in W^{1,2}_{loc}(\Omega; N)$ is an energy minimising map.

Certain properties of tangent maps can be derived from the monotonicity formula. For example all tangent maps are homogeneous degree zero about the origin. We define this property for arbitrary maps, and instead will frequently say such a map is conical, to draw analogies between the tangents in the cases of stationary harmonic maps, mean curvature flows and stationary varifolds.

Definition 3.3.9 (Conical Map)**.** We will say a map $\phi : \mathbb{R}^n \to N$ is conical if $\phi(\lambda x) = \phi(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$.

Remark 3.3.10. Note that equivalently we have that $\phi_{0,\lambda} = \phi$, and so a conical map is its own tangent at the origin.

The following properties are simple consequences of the definition of a conical map.

Proposition 3.3.11 (Derivative and density computations). *Suppose $\phi : \mathbb{R}^n \to N$ is a conical map, which is smooth on* $\mathrm{Reg}(\phi) \subset \mathbb{R}^n$. *Then*

$$\lambda D\phi(\lambda y) = D\phi(y), \quad \textit{for any } \lambda > 0,\, y \in \mathrm{Reg}(\phi).$$

If $\phi : \mathbb{R}^n \to N$ is both conical and stationary harmonic then

$$\Theta_\phi(\lambda y) = \Theta_\phi(y), \quad y \in \mathbb{R}^n.$$

Remark 3.3.12. In the second statement we require that ϕ is stationary harmonic only so that ϕ satisfies the monotonicity formula, in which case we can define the energy density Θ_ϕ.

Proof. The first statement comes from differentiating the definition of a conical map. The second statement follows by using a change of variables in the scaled energy densities, and making use of the first statement. □

Tangent maps to energy minimising maps are conical by the monotonicity formula since we can show the energy density ratios of the tangent map at the origin are independent of the scale.

Lemma 3.3.13 (Tangent Maps are Conical). *Suppose $u \in W^{1,2}(\Omega; N)$ is energy minimising, $x \in \Omega$ and $\phi \in T_x u$. Then ϕ is conical and $\Theta_\phi(0) = \Theta_u(x)$.*

Proof. As ϕ is also energy minimising we may apply the monotonicity formula. By definition let u_{x,λ_i} converge in $W^{1,2}$ to ϕ. By the strong convergence of energy we have that for each $r > 0$

$$\Theta_\phi(0, r) = \lim_{i\to\infty} \Theta_{u_{x,\lambda_i}}(0, r) = \lim_{i\to\infty} \Theta_u(x, \lambda_i r) = \Theta_u(x).$$

In the limit as $r \to 0$ this implies $\Theta_\phi(0) = \Theta_u(x)$ as required. Further by the monotonicity formula Theorem 3.2.2 we have that for any $r > 0$ the following holds.

$$\int_{B_r} \left|\frac{\partial\phi}{\partial\rho}\right| \rho^{2-n}\, \mathrm{d}x = 0.$$

This implies that ϕ is conical since the radial derivative of ϕ is then 0 almost everywhere. □

The tangents to a map $u \in W^{1,2}_{loc}(\Omega; N)$ at a regular point $x \in \mathrm{Reg}(u)$ are constant maps, and as such the energy density at a regular point is 0.

Proposition 3.3.14 (Tangents at regular points)**.** *Let $u \in W^{1,2}_{loc}(\Omega; N)$ and suppose $x \in \mathrm{Reg}(u) \cap \Omega$. Then there is a unique tangent map $\phi \in T_x u$ defined by $\phi(y) = u(x)$ for all $y \in \mathbb{R}^n$. Further the energy density $\Theta_u(x)$ is well defined and equal to 0.*

Remark 3.3.15. Typically we only make use of this for stationary harmonic or energy minimising maps.

Proof. By definition there is $r > 0$ such that u is smooth on $B_r(x) \subset\subset \Omega$. Let $\lambda_i > 0$ denote a null sequence. It suffices to show u_{x,λ_i} converge uniformly in C^1 to the constant map $u(x)$ on compact subsets. Let $K \subset\subset \mathbb{R}^n$ be an arbitrary compact subset. Since K is compact there is $I = I(K) > 0$ such that $i \geq I$ implies $x + \lambda_i y \in B_r(x)$ for any $y \in K$. Once $i \geq I$ we clearly have the following pointwise convergence by continuity of u at x.

$$u_{x,\lambda_i}(y) = u(x + \lambda_i y) \to u(x), \quad \text{for } y \in K, i \geq I.$$

Since K is compact this convergence is in fact uniform. Further for $i \geq I$ and $y \in K$ we have that $Du(x + \lambda_i y)$ is continuous and hence uniformly bounded, implying the following uniform convergence.

$$|Du_{x,\lambda_i}(y)| = \lambda_i |Du(x + \lambda_i y)| \to 0, \quad \text{for } y \in K, i \geq I.$$

As such u_{x,λ_i} converges uniformly in C^1 to $u(x)$ on any compact subset $K \subset\subset \mathbb{R}^n$, implying the constant map $u(x)$ is the unique tangent map to u at x. Now the energy density of u at x is well defined and equal to 0 as follows. Since u is smooth on $B_r(x)$ it follows that $|Du|$ is bounded on $B_{r/2}(x)$, and so

$$\Theta_u(x, \rho) = \rho^{2-n} \int_{B_\rho(x)} |Du| \, \mathrm{d}x \leq \rho^2 \sup_{B_{r/2}(x)} |Du|, \quad \text{for any } \rho < r/2.$$

As such clearly $\Theta_u(x, \rho) \to 0$ as $\rho \to 0$.

□

The regularity theorems for energy minimising and stationary harmonic maps seek to give an inverse statement to Proposition 3.3.14. For the case of energy minimising maps the result is the Schoen-Uhlenbeck regularity theorem, [SU82]. The analogous result for stationary harmonic maps is Bethuel's regularity theorem [Bet93]. In fact the Schoen-Uhlenbeck theorem makes slightly weaker assumptions, so it is useful to state both results to compare the differences.

In the statement of the regularity theorem we make use of the following notation for the average value of a function on a ball.

Definition 3.3.16 (Average Value)**.** Let $u \in W^{1,2}_{loc}(\Omega; N)$ and $B_r(x) \subset\subset \Omega$. Then define

$$U(x, r) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u \, \mathrm{d}\mathcal{L}^n.$$

The following statement of Schoen-Uhlenbeck regularity theorem [SU82] can be found in Simon's book [Sim96b].

Theorem 3.3.17 (Schoen-Uhlenbeck Regularity Theorem)**.** *Let $\Lambda > 0$, $\theta \in (0, 1)$. There is $\epsilon = \epsilon(n, N, \Lambda, \theta) > 0$ such that the following holds. Let $u \in W^{1,2}_{loc}(\Omega; N)$ be an energy minimising map on $B_R(x_0) \subset\subset \Omega$. Suppose u satisfies the following estimate.*

$$R^{-n} \int_{B_R(x_0)} |u - U(x_0, R)|^2 \mathrm{d}\mathcal{L}^n < \epsilon^2, \quad \Theta_u(x_0, R) \leq \Lambda. \tag{3.9}$$

Then u is smooth on $B_{R/4}(x_0)$. Further the following estimates hold for each non-negative integer $j \in \mathbb{N}_0$.

$$R^j \sup_{B_{\theta R}(x_0)} |D^j u| \leq C(j, \Lambda, N, \theta, n) \left(R^{-n} \int_{B_R(x_0)} |u - U(x_0, R)|^2 \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2}}.$$

Remark 3.3.18. Note that (3.9) is implying that u is L^2-close to a constant map on $B_R(x_0)$. This is sufficient for the energy-minimising case, however for the stationary harmonic case we will need to make a stronger assumption.

The following corollary is similar to the regularity theorem for stationary harmonic maps. It follows from the Poincaré inequality, for example see Corollary 1 of section 2.10 in Simon's book [Sim96b].

Corollary 3.3.19 (Schoen-Uhlenbeck Regularity Theorem, Density Version)**.** *There is $\epsilon = \epsilon(n, N) > 0$ such that the following holds. Let $u \in W^{1,2}_{loc}(\Omega; N)$ be an energy-minimising map, and suppose for $B_r(x) \subset\subset \Omega$ we have that*

$$\Theta_u(x, r) < \epsilon. \tag{3.10}$$

Then $x \in \mathrm{Reg}(u)$. Further we have the following estimates

$$\sup_{B_{r/2}(x)} r^j |D^j u| \leq C_j(n, N, \Lambda) \quad \textit{for } j \in \mathbb{N}_0.$$

Remark 3.3.20. Note that (3.10) is implying that $|Du|$ is L^2-close to 0 in some rescaled sense on $B_r(x)$.

The Bethuel regularity theorem [Bet93] for stationary harmonic maps is similar to Corollary 3.3.19. We need the stronger energy density ratio assumption to prove regularity of a stationary harmonic map.

Theorem 3.3.21 (Bethuel Regularity Estimate)**.** *There is $\epsilon_0(n, N) > 0$ and $C = C(n, N) > 0$ such that for any $\epsilon < \epsilon_0$ we have the following. Let $u \in W^{1,2}_{loc}(\Omega; N)$ be a stationary harmonic map and suppose $B_{2r}(x) \subset\subset \Omega$. Further suppose u satisfies the following estimate.*

$$\Theta_u(x, 2r) < \epsilon.$$

Then u is smooth on $B_r(x)$ and we have the following estimate.

$$\sup_{B_r(x)} |Du| \leq Cr^{-1}\sqrt{\epsilon}. \tag{3.11}$$

Remark 3.3.22. Another way to write (3.11) is

$$\sup_{B_r(x)} |Du| \leq Cr^{-1}\sqrt{\Theta_u(x, 2r)},$$

whenever $\Theta_u(x, 2r) < \epsilon_0$.

As a corollary we can show the singular set is locally $\mathcal{H}^{n-2}$ null.

Corollary 3.3.23 (Singular set is locally $\mathcal{H}^{n-2}$-null)**.** *Let $u \in W^{1,2}_{loc}(\Omega; N)$ be stationary harmonic, and $K \subset\subset \Omega$. Then $\mathcal{H}^{n-2}(K \cap \mathrm{Sing}(u)) = 0$.*

Remark 3.3.24. In particular this applies to energy minimising maps, where the compactness allows us to apply this to tangent maps. Later we will see this allows you to rule out the $(n-2)$-dimensional singular strata for energy minimising maps.

Proof. It suffices to show for any $\epsilon > 0$ there is a collection of balls $B_{\rho_j}(y_j)$ covering $\mathrm{Sing}(u)$ with $\sum_j \rho_j^{n-2} < \epsilon$. We know by the regularity theorems above that for each $x \in K \cap \mathrm{Sing}(u)$ we have $\Theta_u(x, r) \geq \epsilon$ for sufficiently small ϵ and r such that $B_r(x) \subset\subset \Omega$. In particular this implies

$$\epsilon r^{n-2} \leq \int_{B_r(x)} |Du|^2 \,\mathrm{d}x, \quad \text{for } x \in K \cap \mathrm{Sing}(u), r < \mathrm{dist}(x, \partial\Omega). \tag{3.12}$$

For any $\delta < \mathrm{dist}(K, \partial\Omega)$ we can pick a maximal integer J such that there exist $y_j \in K \cap \mathrm{Sing}(u)$ with $B_{\delta/2}(y_j)$ pairwise disjoint. By the maximality of J, the balls $B_\delta(y_j)$ must cover $K \cap \mathrm{Sing}(u)$. Now by (3.12) applied to $B_{\delta/2}(y_j)$ we have that

$$J\delta^n \leq 2^n \delta^2 \epsilon^{-1} \int_{B_\delta(K \cap \mathrm{Sing}(u))} |Du|^2 \mathrm{d}x.$$

As such letting $\delta \to 0$ we have that $\mathcal{L}^n(K \cap \mathrm{Sing}(u)) = 0$, and so by dominated convergence theorem we have that

$$\int_{B_\delta(K\cap\mathrm{Sing}(u))} |Du|^2 \,\mathrm{d}x \to 0, \quad \text{as } \delta \to 0.$$

Again by (3.12) it follows that

$$J\delta^{n-2} \le 2^n \epsilon^{-1} \int_{B_\delta(K\cap\mathrm{Sing}(u))} |Du|^2 \,\mathrm{d}x \to 0, \quad \text{as } \delta \to 0.$$

This implies $\mathcal{H}^{n-2}(K \cap \mathrm{Sing}(u)) = 0$. □

3.4 The Singular Set and Stratification

The singular set of an energy minimising map satisfies a well known stratification. That is, we can write the singular set as a union of subsets, each of which has a particular dimension bound. The stratification follows from Federer's dimension reduction argument [Fed69], by studying the spines of tangent maps at singularities. This type of argument was used by Almgren [Alm00] to stratify the singular set of a minimal surface. The arguments are analogous in the case of energy-minimizing and stationary harmonic maps. We make use of Lin's paper [Lin99] for reference to results.

To understand the stratification result we first need to study the spines of conical tangent maps. Recall Definition 3.3.9 of a conical map.

Proposition 3.4.1 (Cones have maximal density at origin)**.** *Suppose $\phi : \mathbb{R}^n \to N$ is conical and stationary harmonic. Then $\Theta_\phi(0) \ge \Theta_\phi(x)$ for any $x \in \mathbb{R}^n$.*

Proof. Let $x \in \mathbb{R}^n$ and $R > |x|$. Recall $\Theta_\phi(0) = \Theta_\phi(0, R)$ by Proposition 3.3.11. Then

$$\Theta_\phi(0) = \Theta_\phi(0, R) \ge \left(1 + \frac{|x|}{R - |x|}\right)^{2-n} \Theta_\phi(x, R - |x|).$$

Now by monotonicity $\Theta_\phi(x, R - |x|) \ge \Theta_\phi(x)$. The result clearly follows letting $R \to \infty$. □

The spine of a stationary harmonic cone can be defined as the set of points where this inequality is an equality.

Definition 3.4.2 (Spine of a Conical Map)**.** Suppose $\phi : \mathbb{R}^n \to N$ is conical and stationary harmonic. Then we define the spine as

$$\mathcal{S}(\phi) = \{x \in \mathbb{R}^n \,:\, \Theta_\phi(x) = \Theta_\phi(0)\}.$$

Another way to define the spine of a stationary harmonic cone is in terms of the translation invariances of the cone.

Proposition 3.4.3. *Suppose $\phi : \mathbb{R}^n \to N$ is conical and stationary harmonic. Then $\phi_{y,s} = \phi$ for any $y \in \mathcal{S}(\phi)$ and $s > 0$, and $\mathcal{S}(\phi) \subset \mathbb{R}^n$ is a linear subspace.*

Proof. It suffices to show $\phi_{y,1} = \phi$ for any $y \in \mathcal{S}(\phi)$, since $\phi_{0,s} = \phi$ for any $s > 0$ by definition of ϕ being conical. Recall that in the proof of Proposition 3.4.1 we used that $\Theta_\phi(y, R) \geq \Theta_\phi(y)$. As such $\Theta_\phi(y) = \Theta_\phi(0)$ implies $\Theta_\phi(y, R)$ is constant in R. By monotonicity Theorem 3.2.2 this implies

$$\int_{B_r} \rho_y^{2-n} \left| \frac{\partial \phi}{\partial \rho_y} \right| \mathrm{d}\mathcal{L}^n = 0, \quad \text{for any } r > 0.$$

As such $\frac{\partial \phi}{\partial \rho_y} = 0$ almost everywhere, implying ϕ is constant along rays from y.

$$\phi_{y,1}(x) = \phi(y + x) = \phi(y + \lambda x) = \phi_{y,\lambda}(x), \quad \text{for any } \lambda > 0.$$

Now choose $\lambda > 0$ such that $\lambda - \lambda^{-1} = 1$. Then

$$\phi(x) = \phi(\lambda x) = \phi(y + (\lambda x - y)) = \phi_{y,1}(\lambda x - y).$$

Since $\phi_{y,1} = \phi_{y,\lambda^{-2}}$ we then have that

$$\phi(x) = \phi_{y,\lambda^{-2}}(\lambda x - y) = \phi(y + \lambda^{-1} x - \lambda^{-2} y).$$

Now using again that $\phi_{0,\lambda} = \phi$ and $\lambda - \lambda^{-1}$ we have that

$$\phi(x) = \phi((\lambda - \lambda^{-1}) y + x) = \phi(y + x) = \phi_{y,1}(x).$$

This proves $\phi_{y,1} = \phi$ for any $y \in \mathcal{S}(\phi)$. We now show $\mathcal{S}(\phi)$ is a linear subspace. Doing the above now with $\lambda - \lambda^{-1} = -1$ we have that $\phi(x + y) = \phi(x - y)$, that is $\phi_{y,1} = \phi_{-y,1}$. As such for any $y \in \mathcal{S}(\phi)$ we have that $-y \in \mathcal{S}(\phi)$. Now for any $a > 0$ we have that $ay \in \mathcal{S}(\phi)$ by the following.

$$\phi_{ay,1}(x) = \phi_{y,1}(x/a) = \phi(x/a) = \phi(x).$$

Finally if $y, z \in \mathcal{S}(\phi)$ we have that $y + z \in \mathcal{S}(\phi)$ by the following.

$$\phi_{y+z,1}(x) = \phi_{y,1}(x + z) = \phi(x + z) = \phi_{z,1}(x) = \phi(x).$$

□

Now given a stationary harmonic conical map $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ we can define the spine dimension $\dim(\mathcal{S}(\phi)) \le n$. In fact by the translation invariance of ϕ along the $\mathcal{S}(\phi)$ we have that $\dim(\mathcal{S}(\phi)) = n$ if and only if ϕ is constant. This allows us to characterise the regular set in terms of spine dimension.

Proposition 3.4.4 (Regular points have maximal spine dimension)**.** *Suppose that* $u \in W^{1,2}_{loc}(\Omega; N)$ *is an energy minimising map, and* $x \in \Omega$*. Then* $x \in \mathrm{Reg}(u)$ *if and only if* $\dim(\mathcal{S}(\phi)) = n$ *for some tangent map* $\phi \in T_x u$*.*

Remark 3.4.5. Such a result that the maximal spine dimension is only attained at regular points is not true for the singular sets of stationary varifolds, or Brakke flows, due to the possibility of higher multiplicity planes.

Proof. If $x \in \mathrm{Reg}(u)$ then by Proposition 3.3.14 we have that there is a unique constant tangent map $\phi \in T_x u$, and so clearly $\mathcal{S}(\phi) = \mathbb{R}^n$ and $\dim(\mathcal{S}(\phi)) = n$. On the other hand if $x \in \Omega$ and there is some $\phi \in T_x u$ with $\dim(\mathcal{S}(\phi)) = n$ then we have that ϕ is constant, and so by Schoen-Uhlenbeck regularity Theorem 3.3.17 we have that $x \in \mathrm{Reg}(u)$. □

This means the singular set of an energy-minimising map $u \in W^{1,2}(\Omega; N)$ can be considered as the set of points such that all tangent maps have spines of dimension at most $n-1$. In fact it can be shown that for energy minimising maps that the spines can only have dimension at most $n-3$.

Proposition 3.4.6 (Spine dimensions of tangents)**.** *For any energy minimising map* $u \in W^{1,2}_{loc}(\Omega; N)$ *we have that* $\dim(\mathcal{S}(\phi)) \le n-3$ *for any tangent map* ϕ *to* u*.*

Proof. This follows directly from Corollary 3.3.23 since compactness of energy minimising maps implies ϕ is also energy minimising. □

In the case of stationary harmonic maps we do not have compactness, so tangent maps aren't well defined in general. However the following result of Lin [Lin99] shows we could expect up to an $(n-2)$-dimensional singular set.

Proposition 3.4.7 (Singular sets of limits)**.** *Let* $u_i \in H_\Lambda(B_1; N)$ *converge weakly in* $W^{1,2}$ *to* $u \in W^{1,2}(B_1; N)$*. Let* $\epsilon_0 = \epsilon_0(n, N)$ *denote the* ϵ_0 *of Theorem 3.3.21. Define*

$$\Sigma = \bigcap_{r>0} \left\{ x \in B_1 \,:\, \liminf_{i\to\infty} r^{2-n} \int_{B_r(x)} |Du_i|^2 \,\mathrm{d}x \ge \epsilon_0 \right\}.$$

Then $\Sigma \subset B_1$ *is closed and* $\mathcal{H}^{n-2}(\Sigma \cap B_{1/2}) \le C(\epsilon_0, \Lambda, N)$*.*

Proof. We can show Σ is closed in B_1 by showing $B_1 \setminus \Sigma$ is open. This follows by applying Theorem 3.3.21 to the u_i at any point $x \in B_1 \setminus \Sigma$.

Now for any $0 < \delta < 1/2$ we can cover Σ by a finite collection of balls $B_{r_j}(x_j)$ with $r_j < \delta$ such that $B_{r_j/2}(x_j)$ are pairwise disjoint. By definition of Σ we then have that

$$(r_j/2)^{2-n} \int_{B_{r_j/2}(x_j)} |Du_i|^2 \, \mathrm{d}x \geq \epsilon_0 \text{ for any } j, \text{and for sufficiently large i.}$$

Using that u_i have energy on B_1 bounded by Λ this implies

$$\mathcal{H}^{n-2}(\Sigma) \leq C(n)\Lambda/\epsilon_0.$$

□

These bounds imply $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n-3$ for energy minimising maps, and $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n-2$ for weak limits of stationary harmonic maps. We will be interested in working with the maximal singular dimension for some class of maps.

Definition 3.4.8 (Maximal Singular Dimension)**.** When working with energy minimising maps the maximal singular dimension $d = d(\Omega, \Lambda, N) \in [0, n-3]$ is the minimal integer such that $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq d$ for each $u \in W^{1,2}_{loc}(\Omega; N)$ that is energy minimising. When working with stationary harmonic maps the maximal singular dimension is $d = d(\Omega, \Lambda, N) \in [0, n-2]$, the minimal integer such that $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq d$ for each $u \in H_\Lambda(\Omega; N)$.

Remark 3.4.9. Note that if in the stationary harmonic case we have that $d < n-2$, then there is a compactness theorem that shows sequences of stationary harmonic maps have subsequences converging strongly in $W^{1,2}$. In this case we do not need to consider weak limits of energy measures.

The dimension reduction argument stratifies the singular set according to the dimension of the spines of the tangents. Currently we can only define this in the energy minimising case, since we haven't shown tangent maps exist for stationary harmonic maps. However later we will discuss an extension of stationary harmonic maps that do admit tangents, and we can define the singular strata in this case similarly.

Definition 3.4.10 (Strata of the Singular Set)**.** Let $u \in W^{1,2}_{loc}(\Omega; N)$ be an energy-minimising map. For $j = 0, \ldots, n$ we define the singular strata as

$$\mathrm{Sing}_j(u) = \{x \in \Omega \, : \, \dim(\mathcal{S}(\phi)) \leq j \text{ for any } \phi \in T_x u\}.$$

Proposition 3.4.11 ($\mathrm{Sing}_d(u) = \mathrm{Sing}(u)$)**.** *Let $u \in W^{1,2}_{loc}(\Omega; N)$ be an energy minimising map. Then*

$$\mathrm{Sing}_d(u) = \mathrm{Sing}_{d+1}(u) = \ldots = \mathrm{Sing}_n(u) = \mathrm{Sing}(u).$$

Proof. This follows directly from the definition of d. Indeed if this were not the case there would be some tangent ϕ to u with $\dim(\mathcal{S}(\phi)) > d$. Since $\mathrm{Sing}(\phi) \subset \mathcal{S}(\phi)$, and ϕ is energy minimising by the compactness theorem for energy minimising maps, this would contradict the definition of the maximal singular dimension d. □

Dimension reduction states that you can bound the Hausdorff dimension of these strata as follows.

Lemma 3.4.12 (Dimension Reduction)**.** *Let $u \in W^{1,2}(\Omega; N)$ be energy minimising. Then for $j = 0, \ldots, n-3$ we have $\dim_{\mathcal{H}}(\mathrm{Sing}_j(u)) \leq j$.*

As such the top dimensional stratum is particularly interesting to study, as if we can say some property holds for $\mathrm{Sing}_d(u)$, then we can say this property holds across $\mathrm{Sing}(u)$, except possibly on an at most $(d-1)$-dimensional subset.

3.5 The Top Dimensional Singular Set

We are interested in the case that a tangent map has maximal spine dimension across all possible tangent maps in the energy minimising class. Later we will also consider this for stationary harmonic maps and their generalised tangents. For energy minimising maps this dimension is at most $n-3$, but may be less for particular target manifolds. In fact later we will see that we only need this dimension to be maximal among the spine dimensions of all limit maps of some fixed energy minimising map.

Definition 3.5.1 (Cylindrical Map)**.** Given a map $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ which is energy minimising and conical, we say ϕ is cylindrical if it has maximal spine dimension among all tangents to energy minimising maps $u \in W^{1,2}_{loc}(\Omega; N)$. That is $\dim(\mathcal{S}(\phi)) = d$, where d is the maximal spine dimension for energy minimising maps of Definition 3.4.8.

Lemma 3.5.2 (Regularity of Cylindrical Maps)**.** *Given a cylindrical energy minimising map $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ we have that $\mathrm{Sing}(\phi) = \mathcal{S}(\phi)$.*

Remark 3.5.3. In fact we only need that $\dim(\mathcal{S}(\phi))$ is maximal among the spine dimensions of all tangents to ϕ.

Proof. This follows from the maximality of $\dim(\mathcal{S}(\phi))$. Indeed if there were $x \in \mathrm{Sing}(\phi) \setminus \mathcal{S}(\phi)$ let $\psi \in T_x\phi$ be a tangent at that singularity. We show ψ has $d+1$ dimensional spine, contrary to the maximality of d. Let $\lambda_j > 0$ be a null sequence such that ϕ_{x,λ_j} converge to ψ, and recall $\Theta_{\phi_{x,\lambda_j}}(y) = \Theta_\phi(x + \lambda_j y)$. Then for any $y \in \mathcal{S}(\phi)$, we have by upper semi-continuity of the energy density that

$$\Theta_\psi(y) \geq \limsup_{j\to\infty} \Theta_\phi(x + \lambda_j y) = \Theta_\phi(x) = \Theta_\psi(0).$$

The penultimate equality follows as $y \in \mathcal{S}(\phi)$. As such $\mathcal{S}(\phi) \subset \mathcal{S}(\psi)$. However we also have that since ϕ is conical

$$\Theta_\psi(x) \geq \limsup_{j\to\infty} \Theta_\phi((1 + \lambda_j)x) = \Theta_\phi(x).$$

Hence $\Theta_\psi(x) \geq \Theta_\phi(x) = \Theta_\psi(0)$, implying $x \in \mathcal{S}(\psi)$. Since $x \in \mathcal{S}(\psi) \setminus \mathcal{S}(\phi)$, this would imply $\mathcal{S}(\psi)$ is at least $d+1$ dimensional, contrary to the maximality of d.

$\square$

Chapter 4

Weak Limits of Energy Measures

4.1 The class $\mathcal{M}$

Stationary harmonic maps do not form a compact class, even if uniform energy bounds are assumed. Given a sequence of stationary harmonic maps $u_i \in H_\Lambda(\Omega; N)$, Rellich compactness Theorem 2.1.15 implies a subsequence converges strongly in L^2 and weakly in $W^{1,2}$. However this is not enough to guarantee the limit map solves the stationary harmonic differential equation (3.3). We could define tangent maps by these weak limits, but as they are not necessarily stationary harmonic we don't have the monotonicity formula. This is problematic as the monotonicity formula is a central tool for studying the tangent maps.

The following example is given by Lin [Lin99] that demonstrates a particular issue with stationary harmonic maps, in particular a regularity result like Theorem 3.3.17, and compactness result for stationary harmonic maps cannot hold. Suppose $v : S^2 \to S^2$ is conformal. Take the inverse stereographic projection of the domain S^2 to acquire a map $u : \mathbb{R}^2 \to S^2$ which is harmonic with finite energy. The blow up sequence $u_\lambda(x) = u(\lambda x)$ for $\lambda \to \infty$ converge in $W^{1,2}$ to a constant, however the measures $|Du_\lambda|^2 \mathrm{d}x$ converge as Radon measures to $8\pi k \delta_0$ where $k = \deg(v)$ and δ_0 is the Dirac delta measure at the origin. Extending this to $\mathbb{R}^n$ by $\hat{u}(x) = u(x_1, x_2)$ we find that $|D\hat{u}_\lambda|^2 \mathrm{d}x$ converge as Radon measures to an $(n-2)$-dimensional Hausdorff measure supported on an $(n-2)$-dimensional plane. This is very different to the energy minimising case, where u_λ converging to a constant implies $|Du_\lambda|$ converges to 0.

To deal with the lack of compactness in $W^{1,2}$ we make use of the compactness of Radon measures by considering the energy measures $|Du|^2 \mathrm{d}x$ associated to a stationary harmonic map $u \in H_\Lambda(\Omega; N)$. In this case we can extend monotonicity and hence many other useful properties to the limits. We consider the class $\mathcal{M}$ of all weak limits of sequences of such measures. This class was defined and used by Lin [Lin99].

Note that if one is interested purely in the case of energy-minimising maps then the measures in the following can always be thought of as $\mu = |Du|^2 \mathrm{d}x$ for some energy minimising map. In fact in that case the measure theoretic extension isn't necessary to prove the structure result. Similarly this is true for any subclass of stationary harmonic maps for which we have compactness with respect to strong $W^{1,2}$-convergence.

From this point on it is convenient to follow Lin [Lin99] and work on the domain $B_1 \subset\subset \Omega = B_{1+\delta_0}$, where $\delta_0 > 0$ is some fixed small constant. The purpose of $B_{1+\delta_0}$ is to ensure $\mathrm{dist}(x, \partial\Omega) \geq \delta_0$ is uniformly bounded below for all $x \in B_1$. This allows us to define rescales at any point $x \in B_1$ at some uniformly small scale, and also allows us to control the energy of these rescales by an energy bound on the original measure. Since the main results are local in nature this doesn't cause any loss in generality. Indeed if $u \in W^{1,2}_{loc}(\Omega; N)$ and $x \in \Omega$, we can find some ball $B_r(x) \subset\subset \Omega$. Then the rescale $u_{x,r}$ is well defined on B_1, and for slightly smaller $s < r$ we would have $u_{x,s}$ is defined on $B_{1+\delta_0}$. Any result can then be translated and scaled back to u on $B_r(x) \subset\subset \Omega$.

Definition 4.1.1 (The class $\mathcal{M}$)**.** Let $\delta_0 > 0$. Then $\mathcal{M}$ denotes the class of all Radon measures μ on B_1 that arise as weak limits of $|Du_i|^2 \mathrm{d}x$ for a sequence $u_i \in H_\Lambda(B_{1+\delta_0})$, where $\Lambda = \Lambda(\mu) > 0$.

Remark 4.1.2. The $\delta_0 > 0$ is so that points in B_1 maintain some uniform distance from the boundary. For example if $u \in H_\Lambda(B_{1+\delta_0})$ then we can define $u_{y,s}$ for any $y \in B_1$ once s is uniformly small depending only on δ_0.

Note that $\Lambda(\mu) > 0$ is bounded below by $\sup_{r\in(0,1)} \mu(B_r)$, since if $u_i \in H_\Lambda(B_{1+\delta_0}; N)$ and $|Du_i|^2 \mathrm{d}x \to \mu$, then for some non-negative test function $\phi \in C^0_c(B_1)$, with $\phi \leq 1$ and equal to 1 on B_r, we have that

$$\mu(B_r) \leq \mu(\phi) = \lim_{i\to\infty} \int_{B_1} |Du_i|^2 \phi \,\mathrm{d}x \leq \Lambda.$$

Since there is no a priori bound on $\Lambda(\mu)$ we will have to assume such a uniform bound to apply compactness of Radon measures. Fortunately we will see later that such a bound exists for the rescales $\mu_{x,\lambda}$ for $x \in B_1$ and $\lambda > 0$ sufficiently small depending only on δ_0.

It can be shown that measures $\mu \in \mathcal{M}$ are the sum of the energy measure of some limit map and a defect measure. In particular we do not know the limit map is stationary harmonic, though it is approximated in L^2 and weakly in $W^{1,2}$ by a sequence of stationary harmonic maps, and so it is weakly harmonic and satisfies (3.2).

Proposition 4.1.3 (Existence of Associated Maps). *For any $\mu \in \mathcal{M}$ there is $u \in W^{1,2}(B_1; N)$ with $|Du|_{L^2(B_1)} \leq \Lambda = \Lambda(\mu)$ and a non-negative Radon measure $\widetilde{\mu}$ on B_1 such that.*

$$\mu = |Du|^2 \,\mathrm{d}x + \widetilde{\mu}.$$

Further there exists $u_i \in H_\Lambda(B_1; N)$ such that u_i converge to u strongly in L^2, and weakly in $W^{1,2}$, and so that $|Du_i|^2\mathrm{d}x$ converge as Radon measures to μ.

Remark 4.1.4. We call the measure $\widetilde{\mu}$ the defect measure. The support of $\widetilde{\mu}$ corresponds to regular points of u for which the energy of the u_i is not disappearing at small scales. We will define this later.

Note that u is in fact defined and approximated by u_i on $B_{1+\delta_0}$, however we mostly need to think of this map as a map on B_1.

Proof. By definition of $\mathcal{M}$ there is a sequence $u_i \in H_\Lambda(B_{1+\delta_0}; N)$ such that the energy measures $|Du_i|^2\mathrm{d}x$ converge weakly as Radon measures to μ. By Rellich compactness Theorem 2.1.15 we can find a subsequence such that u_i converge strongly in L^2 and weakly in $W^{1,2}$ to some $u \in W^{1,2}(B_1; N)$. Then by Fatou's Lemma 2.1.21 the Radon measure defined by $\mu - |Du|^2\mathrm{d}x$ is non-negative. $\square$

By Proposition 4.1.3 each $\mu \in \mathcal{M}$ has a map $u \in W^{1,2}(B_1; N)$ and a sequence $u_i \in H_\Lambda(B_1; N)$ associated to it. These are not necessarily unique. However it will be useful to have a shorthand notation for when u is a map associated to $\mu \in \mathcal{M}$ in the manner of Proposition 4.1.3.

Definition 4.1.5 (Associated Maps). Given $\mu \in \mathcal{M}$ we say $u \in W^{1,2}(B_1; N)$ is associated to μ if there exists a sequence $u_i \in H_\Lambda(B_1; N)$ such that the energy measures $|Du_i|^2\mathrm{d}x$ converge weakly as measures to μ, u_i converge to u strongly in L^2, and weakly in $W^{1,2}$. If we write (μ, u) as a pair we mean u is a map associated to μ in this sense.

Remark 4.1.6. There is not necessarily a unique u associated to μ. For example if $|Du_i|^2\mathrm{d}x$ converge to μ, then so do $|Dv_i|^2\mathrm{d}x$, where $|Dv_i| = |Du_i|$. Of course one would require v_i to also remain stationary harmonic, with image on N.

Lin [Lin99] shows in Lemma 1.7 that $\mathcal{M}$ is closed under weak limits of a sequence of measures $\mu_i \in \mathcal{M}$. The proof is a diagonal argument. Note that we need a uniform bound on $\Lambda(\mu_i)$ so that we can fix some $\Lambda > 0$ for the limit measure.

Proposition 4.1.7 ($\mathcal{M}$ is closed). *Suppose $\mu_i \in \mathcal{M}$ converge weakly as measure to a Radon measure μ, and $\Lambda(\mu_i)$ are uniformly bounded. Then $\mu \in \mathcal{M}$.*

Proof. Let $\Lambda(\mu_i) \leq \Lambda$ for all i. Since $\mu_i \in \mathcal{M}$ there are $u_j^i \in H_\Lambda(B_{1+\delta_0}; N)$ such that $|Du_j^i|^2 \mathrm{d}x$ converge to μ_i as $j \to \infty$. The diagonal sequence $u_i^i \in H_\Lambda(B_{1+\delta_0}; N)$ then has energy measures $|Du_i^i|^2 \mathrm{d}x$ that converge to μ. $\square$

Corollary 4.1.8 (Compactness of $\mathcal{M}$)**.** *Let $\mu_i \in \mathcal{M}$ with $\Lambda(\mu_i) \leq \Lambda$. Then there is a subsequence of μ_i that converge as Radon measures to some $\mu \in \mathcal{M}$.*

Proof. It suffices to show $\mu_i \to \mu$ for any Radon measure μ by Proposition 4.1.7. This follows directly from the compactness of Radon measures on B_1, since $\sup_i \mu_i(K) \leq \Lambda$ for any $K \subset\subset B_1$. $\square$

We wish to associate a metric to the convergence of both the Radon measure $\mu \in \mathcal{M}$ and an associated map $u \in W^{1,2}$. Recall by Proposition 2.1.23 and Proposition 2.1.26 that we can define a metric d for the weak convergence of Radon measures, and a metric $d_{W^{1,2}}$ for the weak convergence of maps in $W^{1,2}$.

Definition 4.1.9 (Metric on $\mathcal{M}$)**.** Given $\mu, \nu \in \mathcal{M}$ and $u, v \in W^{1,2}(B_1; N)$ associated to μ and ν respectively, we define

$$d_{\mathcal{M}}\left((\mu, u), (\nu, v)\right) = d(\mu, \nu) + |u - v|_{L^2(B_1)} + d_{W^{1,2}}(u, v).$$

Remark 4.1.10. Note that technically we can only metrise the convergence of Radon measures μ with some uniform bound $\mu(K) \leq \Lambda$ for all $K \subset\subset B_1$. However later when we use this metric such a bound will be possible for the measures we consider. In other words the $\Lambda(\mu)$ of Definition 4.1.1 will be uniformly bounded above for the measures we consider later.

This clearly metrises the convergence of the pair $(|Du_i|^2 \mathrm{d}x, u_i)$ to (μ, u) defined by Proposition 4.1.3. We also have compactness of $\mathcal{M}$ under this metric, following from compactness of Radon measures Theorem 2.1.18, the Rellich compactness Theorem 2.1.15, and the fact that $\mathcal{M}$ is closed, Proposition 4.1.7. In the following we show that a sequence of associated maps converge to an associated map, which is a slight alteration of the argument in Proposition 4.1.7.

Proposition 4.1.11 (Associated maps closed under limits)**.** *Suppose $\mu_i \in \mathcal{M}$ and $u_i \in W^{1,2}(B_1; N)$ are associated to μ_i, and $\Lambda(\mu_i) \leq \Lambda$. Further suppose $\mu_i \to \mu \in \mathcal{M}$ and u_i converge strongly in L^2 and weakly in $W^{1,2}$ to $u \in W^{1,2}(B_1; N)$. Then u is associated to μ, and $\mu \in \mathcal{M}$.*

Remark 4.1.12. Of course we have that $d_{\mathcal{M}}((\mu_i, u_i), (\mu, u)) \to 0$.

Once again the assumption $\Lambda(\mu_i) \leq \Lambda$ will generally apply for our purposes since μ_i will be some sequence of rescales of some fixed μ, in which case we shall see that $\Lambda(\mu_i) \leq C\Lambda(\mu)$.

Proof. By definition if u is associated to μ we have that $d_{\mathcal{M}}((\mu, u), (\mu_i, u_i)) \to 0$. So we must show that there is a sequence $v^j \in H_\Lambda(B_1; N)$ such that v^j converge strongly in L^2 and weakly in $W^{1,2}$ to u, and $|Dv^j|^2\mathrm{d}x$ converge to μ weakly as Radon measures. To show this we use a diagonal argument. Since $\Lambda \geq \Lambda(\mu_i)$ there are $v_i^j \in H_\Lambda(B_1; N)$ that converge to (μ_i, u_i). Then for each $i > 0$ there is $J_i > 0$ such that $j \geq J_i$ implies

$$d_{\mathcal{M}}((|Dv_i^j|^2\mathrm{d}x, v_i^j), (\mu_i, u_i)) < \frac{1}{i}.$$

Then we can simply choose the diagonal sequence $v^j = v_j^{J_j}$. By the triangle inequality we have that v^j converges to (μ, u) in the appropriate way. □

Many of the definitions and properties of stationary harmonic maps extend to measures $\mu \in \mathcal{M}$. For example we can extend the energy density ratios as simply the density ratios of the measure μ.

Definition 4.1.13 (Energy Density Ratios on $\mathcal{M}$)**.** Given $\mu \in \mathcal{M}$, $x \in B_1$ and $0 < r < 1 + \delta_0 - |x|$ we define the energy-density ratio of μ at x at scale r as

$$\Theta_\mu(x, r) = r^{2-n}\mu(B_r(x)).$$

Remark 4.1.14. Of course when $\mu = |Du|^2\,\mathrm{d}x$ for a stationary harmonic map $u \in H_\Lambda(B_1; N)$ we have that $\Theta_\mu(x, r) = \Theta_u(x, r)$ for all $0 < r < \frac{1}{1-|x|}$.

Note that the bound $r < 1+\delta_0-|x|$ is needed so that μ is defined on $B_r(x)$, indeed $\mathrm{dist}(x, \partial B_{1+\delta_0}) = 1 + \delta_0 - |x|$. In particular this is uniformly bounded below by δ_0, so energy density ratios are defined at scales $r < \delta_0$ for any $x \in B_1$.

The energy density ratios of measures $\mu \in \mathcal{M}$ are monotonic following directly from the monotonicity formula for stationary harmonic maps Lemma 3.2.2. Recall that $\rho_x(y) = |x - y|$ is the radial distance from x.

Lemma 4.1.15 (Monotonicity on $\mathcal{M}$)**.** *Given $\mu \in \mathcal{M}$ and $x \in B_1$, for $r \in (0, 1+\delta_0 - |x|)$ the energy density ratio $\Theta_\mu(x, r)$ is monotonically non-decreasing as a function of r. In fact if u is an associated map to μ then we have that for any $R > r > 0$ the following holds.*

$$\Theta_\mu(x, R) - \Theta_\mu(x, r) \geq \int_{B_R(x)\setminus B_r(x)} \left|\frac{\partial u}{\partial \rho_x}\right|^2 \rho_x^{2-n}\,\mathrm{d}\mathcal{L}^n. \tag{4.1}$$

Proof. Suppose $u_i \in H_\Lambda(B_{1+\delta_0}; N)$ are such that $|Du_i|^2 \mathrm{d}x$ converge to μ weakly as Radon measures on B_1, and u_i converge strongly in L^2 and weakly in $W^{1,2}$ to u. Then for almost every $0 < r < 1 + \delta_0 - |x|$ we have that

$$\Theta_{u_i}(x, r) \to \Theta_\mu(x, r).$$

As such for almost every every $0 < r < R < 1 + \delta_0 - |x|$ we have that

$$\Theta_\mu(x, R) - \Theta_\mu(x, r) = \lim_{i\to\infty} \Theta_{u_i}(x, R) - \Theta_{u_i}(x, r).$$

By monotonicity formula of Theorem 3.2.2 we then have that

$$\Theta_\mu(x, R) - \Theta_\mu(x, r) \geq \liminf_{i\to\infty} \int_{B_R(x)\setminus B_r(x)} \left|\frac{\partial u_i}{\partial \rho_x}\right|^2 \rho_x^{2-n} \,\mathrm{d}\mathcal{L}^n.$$

Finally by Fatou's Lemma 2.1.21 we have that

$$\Theta_\mu(x, R) - \Theta_\mu(x, r) \geq \int_{B_R(x)\setminus B_r(x)} \left|\frac{\partial u}{\partial \rho_x}\right|^2 \rho_x^{2-n} \,\mathrm{d}\mathcal{L}^n.$$

This holds for almost every $0 < r < R < 1 + \delta_0 - |x|$. Now for arbitrary $0 < r < R < 1 + \delta_0 - |x|$ we can find $r_i \geq r$ and $R_i \leq R$ such that $r_i \to r$, $R_i \to R$ and (4.1) holds for r_i and R_i. Then we have that

$$\mu(B_r(x)) \leq r_i^{n-2}\Theta_\mu(x, r_i) \leq r_i^{n-2}\Theta_\mu(x, R_i) = \left(\frac{r_i}{R_i}\right)^{n-2} \mu(B_{R_i}(x)).$$

As such it follows that since $R_i \to R > 0$ and $r_i \to r > 0$ we have that

$$\Theta_\mu(x, r) \leq \lim_{i\to\infty} \left(\frac{r_i}{r}\right)^{n-2} \left(\frac{R}{R_i}\right)^{n-2} \Theta_\mu(x, R) = \Theta_\mu(x, R).$$

Further one can easily see that

$$\Theta_\mu(x, R) - \Theta_\mu(x, r) \geq \int_{B_R(x)\setminus B_r(x)} \left|\frac{\partial u}{\partial \rho_x}\right|^2 \rho_x^{2-n} \,\mathrm{d}\mathcal{L}^n.$$

□

The upper-semicontinuity of density Lemma 3.2.6 also extends. The proof is exactly as before, relying solely on monotonicity.

Lemma 4.1.16 (Upper-Semicontinuity on $\mathcal{M}$)**.** *Let $\mu_j \in \mathcal{M}$ be a sequence converging weakly as measures to $\mu \in \mathcal{M}$, $x_j \in B_1$ a sequence of points converging to $x \in B_1$, and $r_j > 0$ a null sequence. Then we have the following.*

$$\Theta_\mu(x) \geq \limsup_{j\to\infty} \Theta_{\mu_j}(x_j, r_j).$$

Remark 4.1.17. As before we also have the following result by monotonicity.

$$\Theta_\mu(x) \geq \limsup_{j\to\infty} \Theta_{\mu_j}(x_j).$$

Note that the necessary bound $r_j < 1 + \delta_0 - |x_j|$ eventually holds since $x_j \to x$ and $r_j \to 0$.

4.2 Tangent Measures

The rescales and translations of Definition 3.3.1 also have analogous definitions for the measures $\mu \in \mathcal{M}$. In this case we need to scale both the argument of the measure and the measure itself. For stationary harmonic maps this was carried out automatically by the change of variables formula.

Definition 4.2.1 (Rescales of measures)**.** Given $\mu \in \mathcal{M}$, $y \in B_1$, $s > 0$, for any $A \subset B_{1/s}(-y/s)$ we define

$$\mu_{y,s}(A) = s^{2-n}\mu(y + sA).$$

Remark 4.2.2. For $s < 1 - |y|$ we have that $B_1 \subset B_{1/s}(-y/s)$, so $\mu_{y,s}$ are defined on B_1.

We have the following result by simply applying the translation and rescales to the sequence of maps $u_i \in H_\Lambda(B_1; N)$ such that $|Du_i|^2\mathrm{d}x \to \mu$. One of the reasons for allowing $\Lambda(\mu)$ in Definition 4.1.1 to depend on μ is so that $\mu_{x,s} \in \mathcal{M}$.

Lemma 4.2.3 ($\mathcal{M}$ is closed under rescale and translation)**.** *Given $\mu \in \mathcal{M}$, $x \in B_1$ and $s \in (0, \frac{1+\delta_0-|x|}{1+\delta_0})$ we have that $\mu_{x,s} \in \mathcal{M}$.*

Remark 4.2.4. In particular μ_{x_i,s_i} are eventually in $\mathcal{M}$ for any x_i that converge to $x \in B_1$, and any null sequence $s_i > 0$. Note that $\frac{1+\delta_0-|x|}{1+\delta_0} \geq \frac{\delta_0}{1+\delta_0}$ for all $x \in B_1$.

Proof. By definition there are $u_i \in H_\Lambda(B_{1+\delta_0}; N)$ such that $|Du_i|^2\mathrm{d}x \to \mu$. Now let $v_i = (u_i)_{x,s}$. Clearly v_i are still stationary harmonic, and defined on $B_{(1+\delta_0)/s}(-x/s)$. For $s < \frac{1+\delta_0-|x|}{1+\delta_0}$ this ball contains $B_{1+\delta_0}$. Clearly $|Dv_i|^2\mathrm{d}x$ converge to $\mu_{x,s}$, so we now only need to show the v_i have uniformly bounded energy. To show this note that by a change of variables we have

$$\int_{B_{1+\delta_0}} |Dv_i|^2 \,\mathrm{d}x = \int_{B_{s(1+\delta_0)}(x)} s^{2-n}|Du_i|^2\mathrm{d}x = (1+\delta_0)^{n-2}\Theta_{u_i}(x, s(1+\delta_0)).$$

Now since $|x| < 1$ we have that $s(1+\delta_0) < 1+\delta_0 - |x| < \delta_0$, and $B_{\delta_0}(x) \subset B_{1+\delta_0}$. Then by the fact that $\int_{B_{1+\delta_0}} |Du_i|^2 \,\mathrm{d}x \le \Lambda$ and by monotonicity we have that

$$\Theta_{u_i}(x, s(1+\delta_0)) \le \Theta_{u_i}(x, \delta_0) \le \delta_0^{2-n}\Lambda.$$

Putting this together gives that

$$\int_{B_{1+\delta_0}} |Dv_i|^2 \,\mathrm{d}x \le \left(\frac{1+\delta_0}{\delta_0}\right)^{n-2} \Lambda. \tag{4.2}$$

□

We have the following important bound that follows from (4.2).

Corollary 4.2.5 (Energy bound for rescales)**.** *Let $\mu \in \mathcal{M}$ and suppose for $\Lambda = \Lambda(\mu) > 0$ there is $u_i \in H_\Lambda(B_{1+\delta_0}; N)$ such that $|Du_i|^2\mathrm{d}x \to \mu$ converge as Radon measures on B_1. Then for any $x \in B_1$ and $s \in (0, \frac{1+\delta_0-|x|}{1+\delta_0})$ we can find $v_i \in H_{\widetilde{\Lambda}}(B_{1+\delta_0}; N)$ such that $|Dv_i|^2\mathrm{d}x$ converge as Radon measures to $\mu_{x,s}$ on B_1, where $\widetilde{\Lambda} \le (\frac{1+\delta_0}{\delta_0})^{n-2}\Lambda$.*

Remark 4.2.6. In other words $\Lambda(\mu_{x,s}) \le C(\delta_0, n)\Lambda(\mu)$ for all $x \in B_1$ and s sufficiently small depending only on δ_0.

By the monotonicity Lemma 4.1.15 and the closure of $\mathcal{M}$ under rescales and weak limits we can define tangent measures in $\mathcal{M}$.

Definition 4.2.7 (Limit and Tangent Measures)**.** Given $\mu \in \mathcal{M}$ and $x \in B_1$, a limit measure to μ at x is any measure $\eta \in \mathcal{M}$ that can be achieved as the weak limit of μ_{x_j,s_j} for some sequence $x_j \in B_1$ with $x_j \to x$, and a null sequence $s_j > 0$.

A tangent measure to $\mu \in \mathcal{M}$ at $x \in \Omega$ is any limit measure $\eta \in \mathcal{M}$ that can be achieved as the limit of a sequence μ_{x,s_j} where $s_j > 0$ is a null sequence. Let $T_x\mu$ denote the collection of all tangent measures to μ at x.

Remark 4.2.8. Of course tangent measures are not necessarily unique. Note that by Proposition 4.1.7, Proposition 4.2.3 and Corollary 4.2.5 a limit measure at any point $x \in B_1$ is in $\mathcal{M}$. In particular limit measures obey the monotonicity Lemma 4.1.15.

We can show tangent measures also have a conical property due to monotonicity. As before we will say a measure is conical if it is invariant under dilations.

Definition 4.2.9 (Conical Measure)**.** Given $\eta \in \mathcal{M}$ we say η is conical if $\eta_{0,\lambda} = \eta$ for each $\lambda > 0$.

Remark 4.2.10. Here we are restricting $\eta_{0,\lambda}$ to B_1 when $\lambda < 1$. We can extend a conical η from B_1 to $\mathbb{R}^n$ as follows. The rescales $\eta_{0,\lambda}$ are defined on $B_{1/\lambda}$, and as such for any $K \subset\subset \mathbb{R}^n$ we can define $\eta_{0,\lambda}(K)$ for sufficiently small $\lambda > 0$. Then we defined $\eta(K) = \eta_{0,\lambda}(K)$. Once η is defined on $\mathbb{R}^n$ we can say $\eta = \eta_{0,\lambda}$ on $\mathbb{R}^n$ for any $\lambda > 0$.

Of course if $\phi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ is conical in the sense of Definition 3.3.9 then $|D\phi|^2 \mathrm{d}x$ is a conical measure.

The following is a simple consequence of the definition.

Proposition 4.2.11 (Conical measures have constant density)**.** *Let $\eta \in \mathcal{M}$ be a conical measure. Then $\Theta_\eta(0, r)$ is constant for $r \in (0, 1)$.*

Remark 4.2.12. This is true for all $r > 0$ if we make the extension of η to $\mathbb{R}^n$ as described in Remark 4.2.10.

The converse statement to Proposition 4.2.11 is proved by Lin [Lin99] in Lemma 1.7(ii). Here Lin assumes η is a tangent measure, then shows η is a cone since it has constant density. However the proof follows purely from the fact that the measure has constant density ratios. The idea is to show η is invariant on rescales of annular sets, and to make use of the fact that a map associated to η has radial derivative equal to 0.

Lemma 4.2.13 (Constant Density implies Conical)**.** *Suppose $\eta \in \mathcal{M}$ has energy density ratios $\Theta_\eta(0, r)$ independent of $r \in (0, 1)$. Then η is conical.*

Proof. The aim is to show $\eta_{0,\lambda} = \eta$ on B_1 for each $\lambda \in (0, 1)$. We do this by splitting η into radial and spherical measures, using polar coordinates. Note that it suffices to show that for each Borel subset $A \subset S^{n-1}$ we have the following

$$\left| \frac{\eta(A_{R,\delta})}{R^{n-3}} - \frac{\eta(A_{\rho,\delta})}{\rho^{n-3}} \right| \leq C\delta^2, \tag{4.3}$$

where for each $0 < \delta < \rho < R$ the annular regions $A_{\rho,\delta}$ are defined by

$$A_{\rho,\delta} = \{tx \,:\, x \in A, t \in (\rho - \delta, \rho + \delta)\}.$$

To see this, firstly for any $\lambda > 0$ we have that

$$A_{r,\delta} = \cup_k A_{r+k\lambda\delta,\lambda\delta},$$

where k takes values between $-1/\lambda$ and $1/\lambda$. As such we have by (4.3) that

$$\eta(A_{r,\delta}) = \sum_k \eta(A_{r+k\lambda\delta,\lambda\delta}) = \sum_k \left(\frac{\eta(A_{r,\lambda\delta})}{r^{3-n}} + C\delta^2 \right) (r + k\lambda\delta)^{3-n}.$$

Then we have that since k takes values between $-1/\lambda$ and $1/\lambda$ the following holds.

$$\left|\eta(A_{r,\delta}) - \lambda^{-1}\eta(A_{r,\lambda\delta})\right| \leq \frac{C\delta^2}{\lambda} + \lambda^{-1}\left(\left(1+\frac{\lambda\delta}{r}\right)^{3-n} - 1\right).$$

In other words

$$\left|\eta(A_{r,\delta}) - \lambda^{-1}\eta(A_{r,\lambda\delta})\right| \to 0 \quad \text{as } \delta \to 0 \tag{4.4}$$

Now for any bounded set $K \subset \mathbb{R}^n$ there is a constant $C(K) > 0$ such that we can cover K by $C(K)/\delta$ annular regions of width δ. This is done by taking intersections of K with various radius balls, and rescaling these onto S^{n-1}. Let $A^i_{r_i,\delta}$ denote such a covering of K which is pairwise disjoint, with $i = 1, \ldots, C(K)/\delta$. Then we have that

$$\eta(K) = \eta(\cup_i A^i_{r_i,\delta}) = \sum_i \eta(A^i_{r_i,\delta}).$$

Then by (4.3) the sum can be approximated as follows for any $\lambda > 0$.

$$\eta(K) = \sum_i \left(C\delta^2 r_i^{n-3} + \frac{\eta(A^i_{\lambda r_i,\delta})}{\lambda^{n-3}}\right).$$

Then by (4.4) we can finally approximate

$$|\eta(K) - \eta_{0,\lambda}(K)| \leq C(K)\delta \to 0, \quad \text{as } \delta \to 0.$$

To prove (4.3) it suffices that the density ratios $r^{2-n}\eta(B_r)$ are constant, and for any $\phi \in C^\infty(S^{n-1})$ with $\phi \geq 0$ we have that the following integral

$$\int_{B_{R+\delta}\setminus B_{R-\delta}} \phi(\theta)^2 r^{3-n}\,\mathrm{d}\eta(r,\theta) \tag{4.5}$$

is invariant in $R > 0$ for almost every $\delta \in (0, R)$. Indeed since $r^{2-n}\eta(B_r) = c$ we have that

$$\frac{\eta(B_{R+\delta}) - \eta(B_{R-\delta})}{\delta} \leq C_0 c, \text{for any } 0 < \delta < R. \tag{4.6}$$

Then for any Borel subset $A \subset S^{n-1}$, by taking ϕ in (4.5) to approximate the characteristic function of A, and using (4.6) bound the error, we get (4.3).

To prove (4.5) requires polar coordinates. Let $v_k \in H_\Lambda(B_1; N)$ converge strongly in L^2 and weakly in $W^{1,2}$ to $v \in W^{1,2}(B_1; N)$, and $|Dv_k|\mathrm{d}x^2$ converge weakly as Radon measures to η. By the monotonicity formula for stationary harmonic maps we have that

$$\int_{B_R\setminus B_r} \left|\frac{\partial v_k}{\partial\rho}\right|^2 \rho^{2-n}\,\mathrm{d}x \to R^{2-n}\eta(B_R) - r^{2-n}\eta(B_r) = 0.$$

Then by Fatou's Lemma 2.1.21 it follows that $\partial v/\partial\rho = 0$.

Now let $\phi \in C^\infty(S^{n-1};[0,\infty))$ and $\psi \in C_c^\infty((0,1))$ with $\psi \geq 1$ and $\int_0^1 \psi \mathrm{d}t = 1$. Let $\psi_\epsilon(r) = \psi(r/\epsilon)/\epsilon$. Let $D_{r,\theta}v_k$ denote the derivative in polar coordinates so that

$$|D_{r,\theta}v_k(r,\theta)|^2 = r^2\left|\frac{\partial v_k}{\partial\rho}(r,\theta)\right| + \left|\frac{\partial v_k}{\partial\theta}(r,\theta)\right|^2.$$

For any $a \in (0,\infty)$ and $\epsilon \in (0,a)$ define

$$E(v_k,\phi,a,\epsilon) = \int_0^\infty \int_{S^{n-1}} |D_{r,\theta}v_k(r+a,\theta)|^2\phi(\theta)\psi_\epsilon(r)\,\mathrm{d}\theta\mathrm{d}r.$$

We can compute how E varies in a as follows.

$$E(v_k,\phi,R,\epsilon) - E(v_k,\phi,\rho,\epsilon) = \int_\rho^R \frac{\mathrm{d}}{\mathrm{d}a}E(v_k,\phi,a,\epsilon).$$

By the polar coordinate form of the elliptic equation (3.3) satisfied weakly by v_k, one can compute as in (1.20) of Lin [Lin99] that

$$\begin{aligned}
&E(v_k,\phi,R,\epsilon) - E(v_k,\phi,\rho,\epsilon) = \\
&\int_0^\infty \int_{S^{n-1}} 2(r+a)^2\left|\frac{\partial v_k}{\partial\rho}\right|^2 (r+a,\theta)\phi(\theta)\psi_\epsilon(r)\,\mathrm{d}\theta\mathrm{d}r\Bigg|_{a=\rho}^{a=R} + \\
&\int_0^\infty \int_\rho^R \int_{S^{n-1}} 2(n-2)(r+a)\left|\frac{\partial v_k}{\partial r}\right|^2 (r+a,\theta)\phi(\theta)\psi_\epsilon(r)\,\mathrm{d}\theta\mathrm{d}a\mathrm{d}r - \\
&\int_0^\infty \int_\rho^R \int_{S^{n-1}} 2\frac{\partial v_k}{\partial r}\frac{\partial v_k}{\partial\theta}(r+a,\theta)\phi(\theta)\psi_\epsilon(r)\,\mathrm{d}\theta\mathrm{d}a\mathrm{d}r.
\end{aligned}$$

Letting $\epsilon \to 0$ it follows that for almost every $0 < \rho < R < \infty$ that

$$\int_{S^{n-1}} \phi(\theta)\,\mathrm{d}\sigma_k(R,\theta) - \int_{S^{n-1}} \phi(\theta)\,\mathrm{d}\sigma_k(\rho,\theta) \to 0, \quad \text{as } k \to \infty, \tag{4.7}$$

where

$$r^{n-3}\mathrm{d}\sigma_k\mathrm{d}r = |D_{r,\theta}v_k(r,\theta)|^2 r^{n-3}\mathrm{d}\theta\mathrm{d}r.$$

By the weak convergence of $|Dv_k|^2\mathrm{d}x$ to η we have that

$$\mathrm{d}\sigma_k(r,\theta)\mathrm{d}r \to r^{3-n}\mathrm{d}\eta(r,\theta).$$

However by (4.7) we have that

$$\mathrm{d}\sigma_k(r+a,\theta) = \mathrm{d}\sigma_k((r+a),\theta)\mathrm{d}(r+a) \to r^{3-n}\mathrm{d}\eta(r,\theta), \quad \text{for any } a > 0.$$

As such $\mathrm{d}\eta(r,\theta) = r^{3-n}\mathrm{d}r\mathrm{d}\sigma(\theta)$, where $\mathrm{d}\sigma(\theta)$ is some weak limit of the $\mathrm{d}\sigma_k(\theta)$. In particular $r^{3-n}\mathrm{d}\eta(r,\theta)$ is invariant in r, and so (4.5) holds.

□

In particular this implies tangents are conical.

Corollary 4.2.14 (Tangent measures are cones)**.** *Given $\mu \in \mathcal{M}$ and $x \in B_1$ we have that any $\eta \in T_x\mu$ is conical and $\Theta_\eta(0) = \Theta_\mu(x)$.*

Proof. Let $\mu_{x,\lambda_j} \to \eta$. By definition we have for almost every $r \in (0,1)$ that $\Theta_\eta(0,r) = \lim_{j\to\infty} \Theta_\mu(x, \lambda_j r) = \Theta_\mu(x)$. Now by monotonicity it follows that $\Theta_\eta(0,r) = \Theta_\mu(x)$ for each $r > 0$, and so η is conical by Lemma 4.2.13. Further we have that $\Theta_\eta(0) = \Theta_\mu(x)$. □

Similar to conical maps, conical measures attain maximal density at the origin.

Proposition 4.2.15 (Maximal density at origin for conical measures)**.** *Let $\eta \in \mathcal{M}$ be a conical measure. Then $\Theta_\eta(0) \geq \Theta_\eta(y)$ for any $y \in B_1$.*

Remark 4.2.16. This is true for any $y \in \mathbb{R}^n$ once η is extended to $\mathbb{R}^n$ by Remark 4.2.10.

Proof. For any $y \in B_1$ and null sequence $\lambda_i > 0$ we have that $\lambda_i y \to 0$. For any $r > 0$ we have that

$$\Theta_\eta(\lambda_i y, \lambda_i r) = (\lambda_i r)^{2-n}\eta(B_{\lambda_i r}(\lambda_i y)) = \Theta_{\eta_{0,\lambda_i}}(y, r).$$

However $\eta_{0,\lambda_i} = \eta$ since η is conical. Then by monotonicity and upper semi-continuity Lemma 4.1.16 we have that

$$\Theta_\eta(0) \geq \limsup_{i\to\infty} \Theta_\eta(\lambda_i y, \lambda_i r) = \limsup_{i\to\infty} \Theta_\eta(y, r) \geq \Theta_\eta(y).$$

□

We define the spine of a conical measure in terms of the density as before. In this case it is useful to note that a conical measure $\eta \in \mathcal{M}$ can be extended from B_1 to $\mathbb{R}^n$ as described in Remark 4.2.10. This helps make it clearer that $\mathcal{S}(\eta)$ is a linear subspace of $\mathbb{R}^n$.

Definition 4.2.17 (The Spine of a Tangent Measure)**.** Given a conical measure $\eta \in \mathcal{M}$ we define the spine of η as

$$\mathcal{S}(\eta) = \{y \in \mathbb{R}^n \,:\, \Theta_\eta(y) = \Theta_\eta(0)\}.$$

As before we can show the spine is a linear subspace of $\mathbb{R}^n$, and that η is invariant under translations along this spine. The argument is similar to Lemma 4.2.13.

Proposition 4.2.18 (Translation invariance along spine)**.** *Let $\eta \in \mathcal{M}$ be a conical measure. Then $\mathcal{S}(\eta)$ is a linear subspace of $\mathbb{R}^n$ and $\eta_{y,\lambda} = \eta$ for any $y \in \mathcal{S}(\eta)$ and $\lambda > 0$.*

Both the conical property of the measure and the translation invariance along the spine of a conical measure extends to any map associated to that measure. This follows from monotonicity Lemma 4.1.15, in particular (4.1).

Lemma 4.2.19 (Properties of associated maps)**.** *Suppose $\eta \in \mathcal{M}$ is conical and $\phi \in W^{1,2}(B_1; N)$ is associated to η. Then ϕ is conical and translation invariant along $\mathcal{S}(\eta)$, that is $\phi_{y,\lambda} = \phi$ for any $y \in \mathcal{S}(\eta)$ and $\lambda > 0$.*

Remark 4.2.20. Note that ϕ is conical on B_1, meaning $\phi_{0,\lambda} = \phi$ for $\lambda \in (0, 1)$, where $\phi_{0,\lambda}$ is restricted to B_1. However we can then extend ϕ to $\mathbb{R}^n$ by this relation, so that ϕ is conical in the sense of Definition 3.3.9.

The lemma is useful in the following case. Let $\mu \in \mathcal{M}$ have associated map $u \in W^{1,2}(B_1; N)$ and suppose $\mu_{x,s_j} \to \eta \in T_x\mu$. Then we can find a subsequence so that u_{x,s_j} converge in L^2, and weakly in $W^{1,2}$ to a conical map ϕ that is associated to η, and $\phi_{y,1} = \phi$ for any $y \in \mathcal{S}(\eta)$.

Proof. Since $\Theta_\eta(y, r)$ is independent of r for any $y \in \mathcal{S}(\eta)$ by Proposition 4.2.11 and Proposition 4.2.18, we have by (4.1) of Lemma 4.1.15 that for any $0 < r < R$ the following holds.

$$\int_{B_R(y)\setminus B_r(y)} \left| \frac{\partial \phi}{\partial \rho_y} \right|^2 \rho_y^{2-n} \, \mathrm{d}\mathcal{L}^n = 0.$$

As such ϕ is conical about y, that is $\phi_{y,\lambda} = \phi$ for any $y \in \mathcal{S}(\eta)$ and $\lambda > 0$. $\square$

4.3 Singular Set

Recall that in Proposition 3.4.7 we showed a weak $W^{1,2}$ limit of stationary harmonic maps has a closed locally $\mathcal{H}^{n-2}$-finite set Σ where there energies of the maps in the sequence are not becoming small in the limit. We will use this to define a singular set for measures $\mu \in \mathcal{M}$.

Definition 4.3.1 (Singular set of $\mu \in \mathcal{M}$)**.** Let $\mu \in \mathcal{M}$ and $u_i \in H_\Lambda(B_1; N)$ be stationary harmonic maps with $|Du_i|^2 \mathrm{d}x \to \mu$. Recall $\epsilon_0 = \epsilon_0(n, N)$ denotes the ϵ_0 of Theorem 3.3.21. Then define $\Sigma(\mu)$ by

$$\Sigma(\mu) = \cap_{r>0}\{x \in B_1 \, : \, \liminf_{i\to\infty} r^{2-n} \int_{B_r(x)} |Du_i|^2 \, \mathrm{d}x \geq \epsilon_0\}.$$

As in Lin [Lin99] we can show $\Sigma(\mu)$ is the union of the singular set of a map associated to μ and the defect measure.

Proposition 4.3.2 (Decomposition of $\Sigma(\mu)$)**.** *Let $\mu \in \mathcal{M}$ and suppose there is a map $u \in W^{1,2}(B_1; N)$ associated to μ. Then*

$$\Sigma(\mu) = \operatorname{spt}(\widetilde{\mu}) \cup \operatorname{Sing}(u),$$

where $\widetilde{\mu} = \mu - |Du|^2\, \mathrm{d}x$.

Proof. By definition there are $u_i \in H_\Lambda(B_1; N)$ converging strongly in L^2, weakly in $W^{1,2}$ to u, and such that $|Du_i|^2 \mathrm{d}x \to \mu$. If $x \in B_1 \setminus \Sigma(\mu)$ then one can show u_i converge smoothly local to x by Theorem 3.3.21. As such $x \in B_1 \setminus \Sigma(\mu)$ implies $x \in \operatorname{Reg}(u)$ and $x \in B_1 \setminus \operatorname{spt}(\widetilde{\mu})$, that is $\operatorname{Sing}(u) \cup \operatorname{spt}(\widetilde{\mu}) \subset \Sigma(\mu)$.

Now if $x \in \Sigma(\mu) \cap B_1$ then we have that

$$\Theta_\mu(x, r) \geq \epsilon_0/2, \quad \text{for a.e. } 0 < r < 1 - |x|. \tag{4.8}$$

If $x \in \operatorname{Reg}(u)$ then $\Theta_u(x, r) \leq \epsilon_0/4$ for sufficiently small r. So by (4.8) we have that

$$r^{2-n}\widetilde{\mu}(B_r(x)) \geq \Theta_\mu(x, r) - \Theta_u(x, r) \geq \frac{\epsilon_0}{4}, \quad \text{for sufficiently small } r > 0.$$

This implies $x \in \operatorname{spt}(\widetilde{\mu})$. $\square$

The following equivalent definition is convenient, and follows directly from the definition of the density function and monotonicity.

Definition 4.3.3 (Singular Set - Density Definition)**.** Let $\epsilon_0 = \epsilon_0(n, N) > 0$ denote the $\epsilon_0(n, N)$ of Bethuel's Regularity Theorem 3.3.21. Then for any $\mu \in \mathcal{M}$ we define the singular set

$$\Sigma(\mu) = \{y \in B_1 \,:\, \Theta_\mu(y) \geq \epsilon_0\}.$$

The points $x \in B_1 \setminus \Sigma(\mu)$ are regular in the sense that locally the approximating maps $u_i \in H_\Lambda(B_1; N)$ are regular and converge smoothly.

Lemma 4.3.4 (Regular points)**.** *Let $\mu \in \mathcal{M}$ be the weak limit of $|Du_i|^2 \mathrm{d}x$ for $u_i \in H_\Lambda(B_1; N)$. Further suppose u_i converge strongly in L^2, and weakly in $W^{1,2}$ to $u \in W^{1,2}(B_1; N)$. Then u_i converge smoothly to u locally on $B_1 \setminus \Sigma(\mu)$. As such for any open ball $B \subset\subset B_1 \setminus \Sigma(\mu)$ we have that $\mu \llcorner B = |Du|^2\, \mathrm{d}x \llcorner B$.*

Proof. By definition at any $y \in B_1 \setminus \Sigma(\mu)$ we have that $\Theta_\mu(y) < \epsilon_0$. Then there is sufficiently small $r > 0$ such that $\Theta_\mu(y, r) < \epsilon_0$. Since $\Theta_{u_i}(y, r)$ converge to $\Theta_\mu(y, r)$ for almost every $r \in (0, (1 - |y|))$, this implies that for sufficiently large i we have that $\Theta_{u_i}(y, r) < \epsilon_0$. By monotonicity for u_i this implies $\Theta_{u_i}(y) < \epsilon_0$ for sufficiently large i. Then by Bethuel regularity Theorem 3.3.21 we have that $y \in \text{Reg}(u_i)$ for all sufficiently large i. The result follows from this, since for any $B \subset\subset B_1 \setminus \Sigma(\mu)$ we have for some $\epsilon > 0$ that $\Theta_\mu(y) \leq \epsilon < \epsilon_0$ uniformly for $y \in B$, and so u_i are smooth and converge smoothly on B. □

We can now extend the Bethuel [Bet93] regularity theorem and estimate stated in Theorem 3.3.21 from stationary harmonic maps to measures in $\mathcal{M}$.

Theorem 4.3.5 (Bethuel Regularity on $\mathcal{M}$)**.** *There is $\epsilon_0 = \epsilon_0(n, N) > 0$ and $C = C(n, N) > 0$ such that for any $\epsilon < \epsilon_0$ we have the following regularity estimate. Let $\mu \in \mathcal{M}$ and suppose $u \in W^{1,2}(B_1; N)$ is associated to μ. Then for $0 < r < (1 - |x|)/2$ we have that*

$$\Theta_\mu(x, 2r) < \epsilon \Rightarrow \sup_{B_r(x)} |Du| \leq Cr^{-1}\sqrt{\epsilon}.$$

Proof. If $\Theta_\mu(x, 2r) < \epsilon$ then by applying Theorem 3.3.21 to the u_j we see that u_j converge smoothly to u on $B_{3r/2}(x)$. As such we can apply the estimate of Theorem 3.3.21 to u on $B_{3r/2}(x)$ to imply the result. □

The singular set of a measure $\mu \in \mathcal{M}$ can be stratified similar to the energy minimising case. The maximal singular dimension Definition 3.4.8 extends to $\mathcal{M}$, as shown in Corollary 1.10 of Lin's paper [Lin99].

Lemma 4.3.6 (Existence of maximal singular dimension)**.** *Either $\Sigma(\mu) = \emptyset$ for all $\mu \in \mathcal{M}$ or there exists an integer $d \in [0, n-2]$ such that $\dim_{\mathcal{H}}(\Sigma(\mu)) \leq d$ for each $\mu \in \mathcal{M}$. Further if d is the minimal such integer then we can find a conical measure $\eta \in \mathcal{M}$ with a d-dimensional spine.*

Remark 4.3.7. Note that for our purposes the case $\Sigma(\mu) = \emptyset$ for all $\mu \in \mathcal{M}$ will not occur, since we will be assuming the existence of a singularity for some $\mu \in \mathcal{M}$. Recall that in the energy minimising case we can in fact choose $d \leq n - 3$.

Definition 4.3.8 (Maximal singular dimension)**.** Given a fixed target manifold N, an energy bound $\Lambda > 0$, and supposing there is at least one singular $\mu \in \mathcal{M}$, we will say that the minimal integer $d \in [0, n-2]$ to satisfy Lemma 4.3.6 is the maximal singular dimension.

Remark 4.3.9. The purpose of this definition is to allow for the case $d < n-2$, rather than working under the assumption that $d = n-2$, and prove results based on this maximal singular dimension d. However note that in the case $d \leq n-3$ there is in fact a compactness theorem for stationary harmonic maps, and so each $\mu \in \mathcal{M}$ is actually the energy measure $|Du|^2 \mathrm{d}x$ of some stationary harmonic associated map $u \in H_\Lambda(B_1; N)$.

The following stratification result for $\Sigma(\mu)$ is Corollary 1.12 of Lin's paper [Lin99]. It follows from the abstract dimension reduction argument of Federer [Fed70] and Almgren [Alm00], see also Appendix A of Simon's notes [Sim83b].

Lemma 4.3.10 (Stratification of the singular set)**.** *Let $\mu \in \mathcal{M}$ and let d denote the maximal singular dimension of $\mathcal{M}$. For $j = 0, 1, \ldots, d$ define the singular strata $\Sigma_j(\mu)$ by*

$$\Sigma_j(\mu) = \{x \in \Sigma(\mu) \,:\, \dim(\mathcal{S}(\eta)) \leq j \textit{ for all } \eta \in T_x\eta\}.$$

Then we have that

$$\Sigma(\mu) = \bigcup_{j=0}^{d} \Sigma_j(\mu).$$

Further the strata satisfy $\dim_{\mathcal{H}}(\Sigma_j(\mu)) \leq j$ *for each $j = 0, 1, \ldots, d$.*

4.4 Cylindrical Measures

By a cylindrical measure we will mean a conical measure that has a maximal dimension spine. In general this means the spine dimension is equal to the maximal singular dimension of Definition 4.3.8. However in some cases it will suffice that the spine dimension is maximal across some subclass of $\mathcal{M}$.

Definition 4.4.1 (Cylindrical Measures)**.** Let $\eta \in \mathcal{M}$ be a conical measure. We say η is cylindrical if $\dim(\mathcal{S}(\eta)) = d$, where d is the maximal singular dimension for $\mathcal{M}$.

We have the following regularity result for cylindrical measures.

Lemma 4.4.2 (Regularity of Cylindrical Measures)**.** *Let $\eta \in \mathcal{M}$ be cylindrical. Then $\mathcal{S}(\eta) = \Sigma(\eta)$.*

Remark 4.4.3. In fact it suffices that $\dim(\mathcal{S}(\eta))$ is maximal among all tangents of η.

Proof. Suppose this were not the case, then there is $\eta \in \mathcal{M}$ with $\dim(\mathcal{S}(\eta)) = d$, and $x \in \Sigma(\eta) \setminus \mathcal{S}(\eta)$. Consider any $\eta' \in T_x\eta$ and let $\eta_{x,\lambda_i} \to \eta'$ converge weakly as measures, for a null sequence λ_i. Since η is translation invariant along it's spine we have the following by upper semi-continuity for any $y \in \mathcal{S}(\eta)$.

$$\Theta_{\eta'}(y) \geq \limsup_{i \to \infty} \Theta_{\eta_{x,\lambda_i}}(y) = \limsup_{i \to \infty} \Theta_\eta(x + \lambda_i y) = \Theta_\eta(x) = \Theta_{\eta'}(0).$$

As such $\mathcal{S}(\eta) \subset \mathcal{S}(\eta')$. We now show that $x \in \mathcal{S}(\eta')$, contrary to the fact that $\dim(\mathcal{S}(\eta')) \leq d$. Indeed by upper semicontinuity we have that

$$\Theta_{\eta'}(x) \geq \limsup_{i \to \infty} \Theta_{\eta_{x,\lambda_i}}(x) = \limsup_{i \to \infty} \Theta_\eta((1 + \lambda_i)x).$$

As η is conical we have that $\Theta_\eta((1 + \lambda_i)x) = \Theta_\eta(x)$, and so $\Theta_{\eta'}(x) \geq \Theta_\eta(x) = \Theta_{\eta'}(0)$, implying $x \in \mathcal{S}(\eta')$ and $\dim(\mathcal{S}(\eta')) \geq d + 1$. This is contrary to d being the maximal singular dimension. □

The cylindrical class of $\mathcal{M}$ is locally compact. This is due to upper semi-continuity, a sequence of singularities must converge to a singularity. Recall that to each $\eta \in \mathcal{M}$ there is $\Lambda(\eta)$ such that $\eta(K) \leq \Lambda(\eta)$ for all $K \subset\subset B_1$.

Lemma 4.4.4 (Compactness of the Cylindrical Class)**.** *Suppose $\eta_i \in \mathcal{M}$ are cylindrical and there is $\Lambda > 0$ such that $\Lambda(\eta_i) \leq \Lambda$ for all i. Then there is a subsequence so that $\eta_i \to \eta \in \mathcal{M}$ converge weakly as measures, η is cylindrical, and $\mathcal{S}(\eta_i)$ converge to $\mathcal{S}(\eta)$. If $\Theta_{\eta_i}(0) \to \theta$ then $\Theta_\eta(0) = \theta$.*

Remark 4.4.5. Suppose $\mathcal{C} \subset \mathcal{M}$ is some subclass of measures that is closed under weak limits and such that all conical $\nu \in \mathcal{C}$ have spine dimension bounded by p. Then the above result holds when $\eta_i \in \mathcal{C}$ are conical and $\dim(\mathcal{S}(\eta_i)) = p$.

Note that the uniform bound $\Lambda(\eta_i) \leq \Lambda$ is true if η_i are all limit measures of some fixed $\mu \in \mathcal{M}$. This follows from the uniform bounds of Corollary 4.2.5. In fact it suffices that $\Theta_{\eta_i}(0)$ are uniformly bounded, since by the conical property of η_i this implies the sufficient uniform mass bounds to apply the compactness of Radon measures.

Proof. A subsequence of η_i converge to a Radon measure η by the compactness of Radon measures, since $\Lambda(\eta_i)$ are uniformly bounded. By Proposition 4.1.7 we have that $\eta \in \mathcal{M}$.

By the weak convergence we have that $\Theta_{\eta_i}(0, r) \to \Theta_\eta(0, r)$ for almost every $r > 0$. Since η_i are conical $\Theta_{\eta_i}(0) = \Theta_{\eta_i}(0, r)$, so by choosing a subsequence we can assume that $\Theta_{\eta_i}(0)$ converge to some $\theta \geq 0$. Note that all cylindrical measures are singular at the origin, and so $\Theta_{\eta_i}(0) \geq \epsilon_0$ by definition. So we have that $\theta \geq \epsilon_0$. Now for almost every $r > 0$ we have that $\Theta_\eta(0, r) = \theta$ and as such $\Theta_\eta(0) = \theta > 0$. By monotonicity we have that $\Theta_\eta(0, r) = \theta$ for every $r > 0$ and as such η is conical.

Now we may choose a subsequence such that the subspaces $\mathcal{S}(\eta_i) \subset \mathbb{R}^n$ converge to a subspace $L \subset \mathbb{R}^n$ with $\dim(L) = d$, the maximal singular dimension. For each $x \in L$ we can find $x_i \in \mathcal{S}(\eta_i)$ such that $x_i \to x$, and as such by upper semicontinuity we have that

$$\Theta_\eta(x) \geq \limsup_{i\to\infty} \Theta_{\eta_i}(x_i) = \limsup_{i\to\infty} \Theta_{\eta_i}(0) = \theta = \Theta_\eta(0).$$

As such $L \subset \mathcal{S}(\eta)$. Since $\dim(\mathcal{S}(\eta)) \leq d$ we have that $\mathcal{S}(\eta) = L$ and η is cylindrical. $\square$

We can use this to show the existence of a minimal cylindrical density.

Proposition 4.4.6 (Existence of Minimal Cylindrical Density)**.** *There is $\alpha \geq \epsilon_0$ such that $\Theta_\eta(0) \geq \alpha$ for any cylindrical measure $\eta \in \mathcal{M}$. Further $\alpha = \Theta_\eta(0)$ for some cylindrical measure $\eta \in \mathcal{M}$.*

Proof. Define $\alpha = \inf\{\Theta_\eta(0) : \eta \in \mathcal{M} \text{ is cylindrical.}\}$. Since cylindrical $\eta \in \mathcal{M}$ are necessarily singular at the origin we have that $\alpha \geq \epsilon_0$. Now given any sequence $\eta_i \in \mathcal{M}$ of cylindrical measures with $\Theta_{\eta_i}(0) \to \alpha$, we can choose a subsequence such that η_i converge to some $\eta \in \mathcal{M}$, and by Lemma 4.4.4 η is cylindrical, with $\Theta_\eta(0) = \alpha$. $\square$

Definition 4.4.7 (Minimum Cylindrical Density)**.** Let $\alpha = \alpha(N, \Lambda, n) > 0$ denote the minimal cylindrical density of Proposition 4.4.6.

Chapter 5

The Structure Theorem for Stationary Harmonic Maps

5.1 Overview

In this chapter we will present the key steps towards proving the structure result for measures $\mu \in \mathcal{M}$. We assume that for some given measure $\mu \in \mathcal{M}$ there is a fixed singularity $x \in \Sigma(\mu)$ such that $\Theta_\mu(x)$ is equal to the minimal cylindrical density of Definition 4.4.7. Further we assume that there is a tangent $\eta \in T_x\mu$ with $\dim(\mathcal{S}(\eta)) = d$, the maximal spine dimension of Definition 4.3.8. Finally we assume there is a map $\phi \in W^{1,2}(B_1; N)$ associated to η such that the slices $\phi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ restricted to a sphere are homotopically non-trivial. In this case we can show that there is some radius $r > 0$ such that $B_r(x) \cap \Sigma(\mu)$ satisfies the assumptions of Reifenberg's Theorem 2.2.3. There are a number of steps to show this. Reifenberg's theorem assumes the closed set can be approximated at each point and scale by subspaces of $\mathbb{R}^n$ with the same dimension.

In the case of energy minimising maps one can replace the measures $\mu \in \mathcal{M}$ throughout this chapter by appropriate energy measures $|Du|^2\mathrm{d}x$ of energy minimising maps.

In section 5.2 we show the existence of pseudo-tangent measures that approximate $\mu_{y,s}$ for $y \in B_r(x)$ and $s > 0$. These pseudo-tangent measures will be conical, and therefore have a spine. The spines of these pseudo-tangents are the subspaces we use to approximate the singular set of μ in the sense of Reifenberg's Theorem 2.2.3. The existence of pseudo-tangent measures in $\mathcal{M}$ follows by making use of limit measures and upper semicontinuity of the energy density ratios. There are however two issues, the spines of the pseudo-tangents do not necessarily have the same dimension at different points and scales. The other issue is we can only show the pseudo-tangents exist for points $y \in B_r(x)$ with $\Theta_\mu(y) \geq \Theta_\mu(x)$.

In section 5.3 we resolve the former of these issues. The main result is a rigidity theorem that says if $\eta \in \mathcal{M}$ is a cylindrical measure, and there is some map ψ associated to η such that the slices $\psi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ are homotopically non-trivial, then for any conical measure $\nu \in \mathcal{M}$ with $\Theta_\nu(0) = \alpha$, the minimal cylindrical density, such that ν is close to η, we can push singularities from ψ to a map associated to ν via the homotopy non-triviality. This pushes up the spine dimension of ν to be maximal, and the homotopy non-triviality of ψ is inherited by a map associated to ν. The aim is then to apply this result to the pseudo-tangents to imply the spines of the pseudo-tangents are all maximal dimensional if there is a cylindrical homotopically non-trivial tangent at the fixed singularity $x \in \Sigma(\mu)$. However to apply this we need to be able to change the base point and scale of the pseudo-tangent in a continuous manner.

In section 5.4 we prove such a continuity result for rescaling and translation of μ by scales $s > 0$ and points $y \in B_r(x)$ with $\Theta_\mu(y) \geq \Theta_\mu(x)$.

Finally in section 5.5 we pull together these results to show that the Reifenberg Theorem 2.2.3 can be applied to the closed set of $y \in B_r(x) \cap \Sigma(\mu)$ with $\Theta_\mu(y) \geq \Theta_\mu(x)$. This requires a no-gaps result, which follows from the homotopy non-trivial property and the assumption the $\Theta_\mu(x)$ is the minimal cylindrical density. Finally we can show $\Theta_\mu(y) \geq \Theta_\mu(x)$ is true for all $y \in B_r(x) \cap \Sigma(\mu)$ for sufficiently small $r > 0$ by an iteration argument. We can then state the structure result in a general form for measures $\mu \in \mathcal{M}$. Following this we discuss this general result in the particular cases of energy minimising maps, stationary harmonic maps, and for the particular target manifolds $N = S^2$ and $N = S^3$ for which there are classifications of possible tangent maps to energy minimising maps.

5.2 Pseudo-Tangents

Given a measure $\mu \in \mathcal{M}$, the tangent measures $\eta \in T_x\mu$ arise as limits of μ_{x,λ_i} for null sequences $\lambda_i > 0$. In a sense the conical measure η is an approximation of μ at x at an infinitesimally small scale, that is $\mu_{x,\lambda}$ for small $\lambda > 0$. We want such conical approximations to $\mu_{x,\lambda}$ for any points $x \in B_1$ and any $\lambda > 0$. These approximations will be called pseudo-tangents.

We fix a singular measure $\mu \in \mathcal{M}$, and a point $x \in \Sigma(\mu)$. On a subset $S^+(x) \subset \Sigma(\mu)$ we show that pseudo-tangents exist at points $y \in S^+(x) \cap B_r(x)$, and scales $s < r$ for sufficiently small $r > 0$.

The use of the pseudo-tangent measures is that the spines of these conical measures provide linear subspaces which can be used to approximate $S^+(x)$ local to x. The goal will be to show these linear subspaces approximate $S^+(x)$ in the sense of the Reifenberg approximation condition Definition 2.2.2.

Throughout this section we will use the notation $d \leq n-2$ for the maximal singular dimension, Definition 4.3.6. Note that if $d \leq n-3$ we can use stationary harmonic maps in place of measures in $\mathcal{M}$, since in this case there is a compactness theorem for stationary harmonic maps in the $W^{1,2}$-norm, see Lin [Lin99] Remark 1.11. In particular $d \leq n-3$ is true for energy minimising maps. In this case we can replace the measures $\mu \in \mathcal{M}$ with just the energy minimising or stationary harmonic maps, and we can replace convergence as Radon measures with strong convergence in $W^{1,2}$.

Definition 5.2.1 (The set $S^+(x)$)**.** Given $\mu \in \mathcal{M}$ and $x \in B_1$ define

$$S^+(x) = \{y \in B_1 \,:\, \Theta_\mu(y) \geq \Theta_\mu(x)\}.$$

A simple consequence of upper-semicontinuity of the energy density is that $S^+(x)$ is closed.

Proposition 5.2.2 ($S^+(x)$ is closed)**.** *Given $y_i \in B_1 \cap S^+(x)$ which converge to $y \in B_1$, we have that $y \in B_1 \cap S^+(x)$.*

The lower bound $\Theta_\mu(y_i) \geq \Theta_\mu(x)$ for a sequence $y_i \in S^+(x)$, alongside upper-semicontinuity, is made use of in the proof of the following lemma for limit measures.

Lemma 5.2.3 (Limit Measures on $S^+(x)$ are conical)**.** *Let $\mu \in \mathcal{M}$ and suppose $u \in W^{1,2}(B_1; N)$ is associated to μ. For some $x \in B_1$ suppose we have sequences $y_j \in S^+(x)$, $\lambda_j > 0$ such that $y_j \to x$, $\lambda_j \to 0$. Then there is a conical measure $\eta \in \mathcal{M}$, a map $\phi \in W^{1,2}(B_1; N)$ associated to η, and a subsequence such that*

$$(\mu_{y_j,\lambda_j}, u_{y_j,\lambda_j}) \to (\eta, \phi), \quad \Theta_\eta(0) = \Theta_\mu(x).$$

Remark 5.2.4. Recall that $(\mu_i, u_i) \to (\mu, u)$ means μ_i converge to μ as Radon measures, and u_i converge to u strongly in L^2 and weakly in $W^{1,2}$.

Note that since η is conical and translation invariant along $\mathcal{S}(\eta)$ so is ϕ, by Lemma 4.2.19.

Proof. Since $\Lambda(\mu_{y_j,\lambda_j})$ are uniformly bounded by Proposition 4.2.5, and $\mathcal{M}$ is closed by Proposition 4.1.7 we may find a subsequence μ_{y_j,λ_j} that converge to a limit measure $\eta \in \mathcal{M}$. By upper-semicontinuity Lemma 4.1.16 and that $y_j \in S^+(x)$ we have that

$$\Theta_\eta(0) \geq \limsup_{j\to\infty} \Theta_\mu(y_j) \geq \Theta_\mu(x).$$

However for almost every $R > 0$ we have that $\Theta_\mu(y_j, R\lambda_j) \to \Theta_\eta(0, R)$. Once again by upper-semicontinuity Lemma 4.1.16 we have that

$$\Theta_\mu(x) \geq \limsup_{j\to\infty} \Theta_\mu(y_j, R\lambda_j) = \Theta_\eta(0, R).$$

As such by monotonicity of Θ_η we have that

$$\Theta_\mu(x) \geq \Theta_\eta(0, R) \geq \Theta_\eta(0) \geq \Theta_\mu(x).$$

This implies $\Theta_\eta(0, R) = \Theta_u(x)$ for all $R > 0$ and so η is conical by Lemma 4.2.13, and $\Theta_\eta(0) = \Theta_\mu(x)$.

To find ϕ we use a diagonal argument. By definition of $\mathcal{M}$ there exists some sequence $u^i \in H_\Lambda(B_1; N)$ for which $(|Du^i|^2 \mathrm{d}x, u_i)$ converge to (μ, u). For each fixed j we have that $(|Du^i_{y_j,s_j}|^2 \mathrm{d}x, u^i_{y_j,s_j})$ converge to $(\mu_{y_j,s_j}, u_{y_j,s_j})$. As such for each j there is $I_j > 0$ such that $i \geq I_j$ implies

$$d_{\mathcal{M}}((\mu_{y_j,s_j}, u_{y_j,s_j}), (|Du^i_{y_j,s_j}|^2, u^i_{y_j,s_j})) < \frac{1}{j}.$$

We take the diagonal sequence $v_j = u^{I_j}_{y_j,s_j}$. Choose a subsequence so that u_{y_j,λ_j} converge strongly in L^2 and weakly in $W^{1,2}$ to a map $\phi \in W^{1,2}(B_1; N)$. Then by the triangle inequality we have that

$$d_{\mathcal{M}}((Dv_j, v_j), (\eta, \phi)) \leq d_{\mathcal{M}}((Du_{y_j,\lambda_j}, u_{y_j,\lambda_j}), (\eta, \phi)) + \frac{1}{I_j}.$$

As such v_j also converge strongly in L^2 and weakly in $W^{1,2}$ to ϕ, and $|Dv_j|^2 \, \mathrm{d}x$ converge to η as Radon measures. This implies ϕ is associated to the conical measure η, so by Lemma 4.2.19 we have that ϕ is also conical and translation invariant along $\mathcal{S}(\eta)$. □

This lemma can be quantified using the metric $d_{\mathcal{M}}$ of Definition 4.1.9. Recall that this metric is in fact only well defined on some subspace of $\mu \in \mathcal{M}$ such that $\Lambda(\mu)$ is uniformly bounded. In this case we can ensure such a bound as we fix $\mu \in \mathcal{M}$ and only consider rescales and their limits $\eta \in \mathcal{M}$ which have $\Lambda(\eta) \leq C(n, \delta_0)\Lambda(\mu)$ by Proposition 4.2.5

Corollary 5.2.5 (Existence of Pseudo-Tangents)**.** *For any $\epsilon > 0$, $\mu \in \mathcal{M}$, $u \in W^{1,2}(B_1; N)$ associated to μ, and $x \in \Sigma(\mu)$ there is $r = r(\mu, N, x, \epsilon) > 0$ such that the* *following holds. For any $y \in S^+(x) \cap B_r(x)$ and $s \in (0, r]$ there is a conical measure $\eta \in \mathcal{M}$, with associated map $\phi \in W^{1,2}(B_1; N)$ such that*

$$\Theta_\eta(0) = \Theta_\mu(x), \text{ and } d_{\mathcal{M}}((\mu_{y,s}, u_{y,s}), (\eta, \phi)) < \epsilon.$$

Proof. This follows by contradiction using Lemma 5.2.3. □

Definition 5.2.6 (Pseudo-Tangents)**.** Let $\mu \in \mathcal{M}$ and suppose there is a map $u \in W^{1,2}(B_1; N)$ associated to μ. For $\epsilon > 0$, $x \in B_1$ and $s > 0$ we say the pair (η, ϕ) is an ϵ-pseudo-tangent at x with scale s if the following holds. The measure $\eta \in \mathcal{M}$ is conical, the map $\phi \in W^{1,2}(B_1; N)$ is associated to η, and we have that

$$d_{\mathcal{M}}((\mu_{x,s}, u_{x,s}), (\eta, \phi)) < \epsilon.$$

Remark 5.2.7. Corollary 5.2.5 shows that for any $x \in \Sigma(\mu)$ and any $\epsilon > 0$ there exists ϵ-pseudo-tangents at all points $y \in S^+(x)$ sufficiently close to x and all scales $s > 0$ sufficiently close to 0. Note that pseudo-tangents are not unique in general.

Since the pseudo-tangents are conical they have a spine. To be able to use these spines in a Reifenberg approximation argument, Theorem 2.2.3, we need that the spines all have the same dimension. In the following section we give sufficient conditions on the singularity x to ensure that the pseudo-tangents on $S^+(x)$ local to x are all cylindrical.

5.3 The Rigidity Theorem

In this section we consider a cylindrical $\eta \in \mathcal{M}$ with associated map $\psi \in W^{1,2}(B_1; N)$. Recall by Definition 3.5.1 that cylindrical measures are conical with maximal spine dimension among all possible conical measures $\eta \in \mathcal{M}$. By supposing the slices of the map ψ along $\mathcal{S}(\eta)$ are homotopically non-trivial according to Definition 2.1.28 we can show that any conical measure $\nu \in \mathcal{M}$ that is sufficiently close to η is also cylindrical, essentially by pushing singularities from ψ to a map associated to ν.

Note that by assuming the slice map $\psi\lfloor_{\mathcal{S}(\eta)^\perp, 0} : S^{n-d} \to N$ is homotopically non-trivial, we have that ψ must be singular at each $x \in \mathcal{S}(\eta)$. In other words $\mathcal{S}(\eta) = \mathrm{Sing}(\phi)$. However this doesn't mean we can rule out the defect measure altogether. Of course in the case $d \leq n - 3$ there is no defect measure as in this case we have compactness of stationary harmonic maps in the strong $W^{1,2}$ convergence. As such the $d = n - 2$ case can only happen if the target N admits homotopically non-trivial weakly harmonic 2-spheres.

The first step is to show the singular set of μ actually lies close to the spine of η. This is a standard argument using only monotonicity and upper-semicontinuity of density. We first prove the result in the limit, then make a quantitative version as a corollary, similar to the existence of pseudo-tangents Lemma 5.2.3.

Lemma 5.3.1 (Singular set near spine)**.** *Let $\mu_i \in \mathcal{M}$ converge weakly as Radon measures on B_1 to a cylindrical measure $\eta \in \mathcal{M}$. Then for any $\epsilon > 0$ there is $I = I(n, N, \Lambda, \epsilon) > 0$ such that for any $i \geq I$ we have*

$$\Sigma(\mu_i) \cap B_{1/2} \subset B_\epsilon(\mathcal{S}(\eta)) \cap B_{1/2}.$$

Remark 5.3.2. Of course $B_{1/2}$ can be replaced by any $K \subset\subset B_1$ given I is allowed to depend on $\operatorname{dist}(K, \partial B_1)$.

Proof. Given $x_i \in \Sigma(\mu_i) \cap B_{1/2}$ there is a convergent subsequence such that $x_i \to x \in B_1$. By the definition of the singular set $\Theta_{\mu_i}(x_i) \geq \epsilon_0 > 0$ where ϵ_0 is from Bethuel regularity Theorem 3.3.21. Then by upper semi-continuity Lemma 4.1.16 we have that

$$\Theta_\eta(x) \geq \limsup_{i\to\infty} \Theta_{\mu_i}(x_i) \geq \epsilon_0.$$

This implies $x \in \Sigma(\eta) \cap B_1$. However by Lemma 4.4.2 we have that $\Sigma(\eta) = \mathcal{S}(\eta)$ as η is a cylindrical measure. So $x \in \mathcal{S}(\eta)$. The result is now clear, if $x_i \in \Sigma(\mu_i) \cap B_{1/2}$ but $\operatorname{dist}(x_i, \mathcal{S}(\eta)) > \epsilon$ we could easily derive a contradiction by taking a convergent subsequence of x_i. $\square$

Let d_* denote some metrisation of the weak convergence of Radon measures on B_1 as in Definition 2.1.24. Then we have the following quantitative version of Lemma 5.3.1.

Corollary 5.3.3. *For any $\epsilon > 0$ and $\theta > 0$ there is $\delta = \delta(n, N, \epsilon, \theta) > 0$ such that the following holds. Suppose $\mu, \eta \in \mathcal{M}$ and η is cylindrical. Suppose $\Theta_\eta(0) \leq \theta$ and $d_*(\mu, \eta) < \delta$. Then we have the following inclusion,*

$$\Sigma(\mu) \cap B_{1/2} \subset B_\epsilon(\mathcal{S}(\eta)) \cap B_{1/2}.$$

Remark 5.3.4. An important point here is that δ only depends on η through θ, an upper bound for the energy density $\Theta_\eta(0)$. Later we will be able to fix a θ since $\Theta_\nu(0) = \Theta_\mu(x)$ for the pseudo-tangents ν of Corollary 5.2.5.

Note that we only need the measures to be close here, rather than requiring associated maps to be close too.

Proof. This follows from a contradiction argument with Lemma 5.3.1. Suppose μ_i, η_i satisfy the assumptions of the corollary, and $d_*(\mu_i, \eta_i) \to 0$. However suppose $\Sigma(\mu_i) \cap B_{1/2}$ are not contained in the ϵ-neighbourhood of $\Sigma(\eta_i)$. Since $\theta_{\eta_i}(0) \leq \theta$ implies uniform mass bounds for the η_i we can use the compactness of the cylindrical class, Lemma 4.4.4, and we can choose a subsequence such that η_i converge to some cylindrical measure $\eta \in \mathcal{M}$, and so that $\mathcal{S}(\eta_i) \to \mathcal{S}(\eta)$. By the regularity of cylindrical measures Lemma 4.4.2 we have that $\Sigma(\eta_i) = \mathcal{S}(\eta_i)$, which implies that $\Sigma(\mu_i)$ are eventually not contained in the ϵ-neighbourhood of $\mathcal{S}(\eta) = \Sigma(\eta)$. However $d_*(\mu_i, \eta) \to 0$ by triangle inequality, contradicting Lemma 5.3.1. □

By the regularity result Theorem 4.3.5 we have that for any $x \in \text{Reg}(\eta)$ that $\Theta_\eta(x) = 0$. As such $\Theta_\eta(x, r) \to 0$ as $r \to 0$ for all $x \in \text{Reg}(\eta)$. When $\eta \in \mathcal{M}$ is a cylindrical measure we know that $\text{Reg}(\eta)$ is the complement of the linear subspace $\mathcal{S}(\eta)$ by Lemma 4.4.2. The aim of the next result is to show that $\Theta_\eta(x, r)$ is uniformly small when x is uniformly separated from the singular set and r is uniformly small.

Lemma 5.3.5 (Uniformly Small Energy Ratios)**.** *For any $\epsilon, \tau, \theta > 0$ there is $r = r(\epsilon, \tau, \theta) > 0$ such that if $\eta \in \mathcal{M}$ is cylindrical, $\Theta_\eta(0) \leq \theta$, and $x \in B_{1/2} \setminus B_\tau(\mathcal{S}(\eta))$ then $\Theta_\eta(x, s) < \epsilon$ for any $s \leq r$.*

Proof. Recall that $\Sigma(\eta) = \mathcal{S}(\eta)$ by Lemma 4.4.2. Suppose the result were false for some $\epsilon, \tau > 0$. Then there is a null sequence r_i, a sequence $\eta_i \in \mathcal{M}$ of cylindrical measures, and $x_i \in \text{Reg}(\eta_i) \cap B_{1/2}$ such that the following holds.

$$\text{dist}(x_i, \mathcal{S}(\eta_i)) \geq \tau, \quad \Theta_{\eta_i}(x_i, s_i) \geq \epsilon \text{ for some } s_i \leq r_i.$$

By compactness of the cylindrical class Lemma 4.4.4 we can choose a subsequence such that η_i converge weakly as measures to a cylindrical measure $\eta \in \mathcal{M}$, $\mathcal{S}(\eta_i)$ converge to $\mathcal{S}(\eta)$, and x_i converge to $x \in B_1$ with $\text{dist}(x, \mathcal{S}(\eta)) \geq \tau$. Since $\Sigma(\eta) = \mathcal{S}(\eta)$ this implies that $x \in \text{Reg}(\eta)$. However this contradicts the following fact due to upper-semicontinuity Lemma 4.1.16.

$$\Theta_\eta(x) \geq \limsup_{i \to \infty} \Theta_{\eta_i}(x_i, s_i) \geq \epsilon$$

□

The aim is to use this result with the derivative estimate of the Bethuel regularity Theorem 4.3.5 to achieve pointwise estimates for maps u, ψ associated to μ and η respectively. By Corollary 5.3.3 if μ and η are sufficiently close in the metric d_* corresponding to weak convergence of measures, then μ, η are both regular outside a neighbourhood of $\mathcal{S}(\eta)$. This regularity implies that outside of a neighbourhood of $\mathcal{S}(\eta)$ the measures are exactly equal to the smooth limit of a sequence of regular stationary harmonic maps, as in Lemma 4.3.4. In the following $d_{\mathcal{M}}$ denotes the metrisation of convergence in $\mathcal{M}$ as in Definition 4.1.9.

Theorem 5.3.6 (Pointwise Estimates off the Spine)**.** *For any $\epsilon > 0$, $\tau \in (0, 1/4)$ and $\theta > 0$ there is $\delta = \delta(n, N, \epsilon, \tau, \theta) > 0$ such that the following holds. Suppose $\mu, \eta \in \mathcal{M}$, η is cylindrical with $\Theta_\eta(0) \le \theta$, and $u, \psi \in W^{1,2}(B_1, N)$ are associated to μ, η respectively. Further suppose*

$$d_{\mathcal{M}}((\mu, u), (\eta, \phi)) < \delta.$$

Then one has that

$$\sup\Big\{|u(x) - \psi(x)| \,:\, x \in B_{\frac{1}{2}} \setminus B_\tau(\mathcal{S}(\eta))\Big\} < \epsilon.$$

Remark 5.3.7. Note that we do not make use of the fact that u and ψ are close in some metrisation of the weak convergence of $W^{1,2}$, which follows from $d_{\mathcal{M}}((\mu, u), (\eta, \phi)) < \delta$.

Proof. We will estimate both $u(x)$ and $\psi(x)$ by their averages Definition 3.3.16. First choose $\delta > 0$ sufficiently small so that by Lemma 5.3.3 we have

$$\Sigma(\mu) \cap B_{\frac{3}{4}} \subset B_{\frac{\tau}{2}}(\mathcal{S}(\eta)).$$

Then for $x \in B_{\frac{1}{2}} \setminus B_\tau(\mathcal{S}(\eta))$, since $\tau < 1/2$ we have that

$$B_{\frac{\tau}{2}}(x) \subset B_{\frac{3}{4}} \setminus B_{\frac{\tau}{2}}(\mathcal{S}(\eta)) \subset \mathrm{Reg}(\mu) \cap \mathrm{Reg}(\eta).$$

As such $\mu \llcorner B_{\frac{\tau}{2}}(x) = |Du|^2 \,\mathrm{d}x$ and $\eta \llcorner B_{\frac{\tau}{2}}(x) = |D\psi|^2 \mathrm{d}x$ by Lemma 4.3.4. We use the following notations for the averages

$$U(x, r) = \omega_n^{-1} r^{-n} \int_{B_r(x)} u \,\mathrm{d}\mathcal{L}^n, \quad \Psi(x, r) = \omega_n^{-1} r^{-n} \int_{B_r(x)} \psi \,\mathrm{d}\mathcal{L}^n.$$

By the regularity of u and ψ on $B_{\tau/2}(x)$ we can make the following estimate for any $r < \frac{\tau}{2}$.

$$|u(x) - U(x, r)| \le r \sup_{B_r(x)} |Du|, \quad |\psi(x) - \Psi(x)| \le r \sup_{B_r(x)} |D\psi|. \tag{5.1}$$

By the Hölder inequality we also have the following estimate

$$|U(x, r) - \Psi(x, r)| \le (\omega_n r^n)^{-\frac{1}{2}} \|u - \psi\|_{L^2(B_r(x))}. \tag{5.2}$$

Now recall that $d_{\mathcal{M}}((\mu, u), (\eta, \psi)) < \delta$ implies $\|u - \psi\|_{L^2(B_1)} < \delta$. Combining (5.1) and (5.2) we have the following for $x \in B_{\frac{1}{2}} \setminus B_\tau(\mathcal{S}(\eta))$ and any $r \leq \frac{\tau}{2}$.

$$|u(x) - \psi(x)| \leq r \left(\sup_{B_r(x)} |Du| + \sup_{B_r(x)} |D\psi| \right) + (\omega_n r^n)^{-\frac{1}{2}} \delta. \tag{5.3}$$

We need to fix some r for which the first two terms on the right hand side of (5.3) are small, then choose δ sufficiently small based on this r. We use the Bethuel regularity estimate Theorem 4.3.5 to bound the first two terms on the right hand side of (5.3). The assumption of this theorem is that the energy ratios of μ and η are sufficiently small. We can show the energy density ratios of ψ are uniformly small by Lemma 5.3.5, then extend this to u by the fact that u is close to ψ.

First given $\epsilon > 0$ choose $\hat{\epsilon} < \min((\frac{\epsilon}{2C})^2, \epsilon_0)$ where $\epsilon_0 = \epsilon_0(n, N)$ and $C = C(n, N)$ are from Theorem 4.3.5. Note that $\hat{\epsilon} = \hat{\epsilon}(\epsilon, n, N)$.

Next choose $0 < R(\epsilon, n, N, \tau) < \tau/2$ such that Lemma 5.3.5 applies with R in place of r and $\hat{\epsilon}/2$ in place of ϵ.

Finally choose $\delta = \delta(\epsilon, n, N, \tau) > 0$ so that $(\omega_n (R/2)^n)^{-\frac{1}{2}} \delta < \frac{\epsilon}{2}$ and $R^{2-n}\delta < \frac{\hat{\epsilon}}{2}$.

Then by Lemma 5.3.5 we have that for any $x \in B_{1/2} \setminus B_\tau(\mathcal{S}(\eta))$ the following holds.

$$\Theta_\eta(x, R) < \frac{\hat{\epsilon}}{2} < \epsilon_0. \tag{5.4}$$

Since $d_*(\mu, \eta) < \delta$ we have that

$$\Theta_\mu(x, R) \leq R^{2-n}\delta + \Theta_\eta(x, R) < \hat{\epsilon} \leq \epsilon_0. \tag{5.5}$$

As such we can apply Theorem 4.3.5 to both μ and η at x to obtain

$$\sup_{B_{\frac{R}{2}}(x)} |Du| \leq CR^{-1}\sqrt{\hat{\epsilon}}, \quad \sup_{B_{\frac{R}{2}}(x)} |D\psi| \leq CR^{-1}\sqrt{\hat{\epsilon}}$$

By substituting this into (5.3), and by the choices of $\hat{\epsilon}$ and δ the result follows for any $x \in B_{\frac{1}{2}} \setminus B_\tau(\mathcal{S}(\psi))$.

$$|u(x) - \psi(x)| \leq C\sqrt{\hat{\epsilon}} + (\omega_n (R/2)^n)^{-\frac{1}{2}} \delta < \epsilon.$$

□

Note that we used that $d_*(\mu, \eta) < \delta$ to imply (5.5), however a similar comparison can be made using that $|u - \psi|_{L^2(B_1)} < \delta$, since μ, η are equal to $|Du|^2 \mathrm{d}x, |D\psi|^2 \mathrm{d}x$ on $B_R(x)$.

Recall by Proposition 2.1.32 there is a $\tau > 0$ such that if two maps on spheres are pointwise separated by at most τ, then the two maps are homotopic. We can now ensure such a pointwise estimate by Theorem 5.3.6. First we need to define slice maps by taking cross sections of $\mathbb{R}^n$ along $\mathcal{S}(\eta)$.

Definition 5.3.8 (Slice Map)**.** Suppose $u \in W^{1,2}(B_1; N)$ and $L \in G_k(n)$ is a k-dimensional plane. Then for each $x \in L$ we define a slice map $u\lfloor_{L^\perp,x} : L^\perp \to N$ by $u\lfloor_{L^\perp,x}(y) = u(x+y)$.

For any $\tau > 0$ and $x \in L$ define $C_r(x, L^\perp) \subset L^\perp$ as the sphere of radius r centred on x in $L^\perp$, and $D_r(x, L^\perp)$ as the disk of radius r centred on x in $L^\perp$. Note that $C_r(x, L^\perp) = \partial D_r(x, L^\perp)$.

Remark 5.3.9. We may occasionally refer to a slice map $u\lfloor_{L^\perp,x}$ by simply u_x when it is clear what L is, and that there is no possibility of confusing this with the translated map $u_{x,1}$.

One can also think of the slice map as a map on the set $\{x\} \times L^\perp$ for each $x \in L$, which gives the idea of slicing $\mathbb{R}^n$ along a subspace L. Further if one chooses coordinates by taking an orthonormal basis of L as the first k coordinates, and an orthonormal basis on $L^\perp$ as the next $n-k$ coordinates, we can think of $u_x(y) = u(x, y)$.

If $\psi \in W^{1,2}(B_1; N)$ is associated to a conical measure $\eta \in \mathcal{M}$ then by translation invariance Lemma 4.2.19 the slices ψ_x along $\mathcal{S}(\eta)$ are independent of $x \in \mathcal{S}(\eta)$. Further the restriction of ψ_x to a sphere is independent of the radius of that sphere.

Proposition 5.3.10 (Slices of conical maps)**.** *Let $L \in G_k(n)$ and suppose that the map $\psi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ satisfies $\psi_{y,\lambda} = \psi$ for all $\lambda > 0$ and all $y \in L$. Then $\psi\lfloor_{L^\perp,y} = \psi\lfloor_{L^\perp,0}$ for all y in L. Further letting $\widetilde{\psi}_r = \psi\lfloor_{L^\perp,0} : C_r(y, L^\perp) \to N$, we have that $\widetilde{\psi}_r(rx) = \widetilde{\psi}_1(x)$ for all $r > 0$ and $x \in C_1(y, L^\perp)$.*

Remark 5.3.11. Note that a conical map $\psi \in W^{1,2}(B_1; N)$ associated to a conical measure $\eta \in \mathcal{M}$ can easily be extended to a map $\psi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$. So in particular the proposition applies to the maps associated to conical measures $\eta \in \mathcal{M}$ with $L = \mathcal{S}(\eta)$.

Proof. The claim that $\psi\lfloor_{L^\perp,y}$ is independent of $y \in L$ follows immediately from the translation invariance of ψ along L. The claim that $\widetilde{\psi}_r$ is independent of r follows immediately from the fact that $\psi_{0,r} = \psi$. $\square$

Corollary 5.3.12 (Homotopy type of slices)**.** *Let $L \in G_k(n)$ and consider a map $\psi \in W^{1,2}_{loc}(\mathbb{R}^n; N)$ satisfying $\psi_{y,\lambda} = \psi$ for all $y \in L$ and $\lambda > 0$. Then the homotopy type of $\psi\lfloor_{L^\perp,y}$ restricted to $C_r(y, L^\perp)$ is independent of r.*

We now aim to compare the associated maps of two close measures $\mu, \eta \in \mathcal{M}$ when η is cylindrical. In particular we compare the topological properties of the slices of the associated maps along $\mathcal{S}(\eta)$. In the following we use d to denote the maximal singular dimension, Definition 4.3.8

Corollary 5.3.13 (Homotopy Equivalence of Slices)**.** *For any $\tau > 0$ and $\theta > 0$ there is $\delta = \delta(n, N, \tau, \theta) > 0$ such that the following holds. Suppose $\mu, \eta \in \mathcal{M}$, η is cylindrical with $\Theta_\eta(0) \leq \theta$, and $u, \psi \in W^{1,2}(B_1; N)$ are associated to μ, η respectively. Further suppose $d_{\mathcal{M}}((\mu, u), (\eta, \psi)) < \delta$. Then the slice map along $\mathcal{S}(\eta)$ defined by $u\lfloor_{\mathcal{S}(\eta)^\perp, x}$ restricted to $C_{1/2}(x, \mathcal{S}(\eta)^\perp)$ is homotopically equivalent to $\psi\lfloor_{L^\perp, 0}$ restricted to $C_{1/2}(0, \mathcal{S}(\eta)^\perp)$ for each $x \in \mathcal{S}(\eta)$.*

If $\psi\lfloor_{\mathcal{S}(\eta)^\perp, 0}$ is further assumed to be homotopically non-trivial when restricted to $C_{1/2}(0, \mathcal{S}(\eta)^\perp)$ then $D_{1/2}(x, \mathcal{S}(\eta)^\perp) \cap \mathrm{Sing}(u)$ is non-empty for each $x \in B_{1/2} \cap \mathcal{S}(\psi)$. As such $\mathrm{Sing}(u)$ and consequently $\Sigma(\mu)$ have Hausdorff dimension d.

Remark 5.3.14. Note that since the homotopically non-trivial property is a property of the associated maps, we can only use this to show the existence of singularities of u. It may be possible to show $\Sigma(\mu)$ is at least d-dimensional by making use of the supports of the defect measures also.

Proof. The first part is clear as Theorem 5.3.6 implies that once δ is sufficiently small then u is sufficiently close to ψ pointwise on $C_{1/2}(x, \mathcal{S}(\eta)^\perp)$ so that Proposition 2.1.32 applies to the slice maps. Note that the slices of ψ at $x \in \mathcal{S}(\eta)$ are independent of x by Proposition 5.3.12, and so $u\lfloor_{\mathcal{S}(\eta)^\perp, x}$ restricted to $C_{1/2}(x, \mathcal{S}(\eta)^\perp)$ is in fact homotopically equivalent to $\psi\lfloor_{\mathcal{S}(\eta)^\perp, 0}$ restricted to $C_r(0, \mathcal{S}(\eta)^\perp)$ for any $r > 0$.

For the second part, if $D_{1/2}(x, \mathcal{S}(\eta)^\perp) \cap \mathrm{Sing}(u)$ was empty, the slice $u\lfloor_{\mathcal{S}(\eta)^\perp, x}$ would be homotopically trivial on $C_{1/2}(x, \mathcal{S}(\eta)^\perp)$. This would contradict the assumption that ψ has homotopically non-trivial slices. Since $D_{1/2}(x, \mathcal{S}(\eta)^\perp)$ contains a singularity of u for each $x \in \mathcal{S}(\eta) \cap B_{\frac{1}{2}}$ and $\dim(\mathcal{S}(\eta)) = d$ it follows that $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \geq d$. Finally $\dim_{\mathcal{H}}(\Sigma(\mu)) \geq \dim_{\mathcal{H}}(\mathrm{Sing}(u)) = d$, however d is the maximal singular dimension for μ, so $\dim_{\mathcal{H}}(\Sigma(\mu)) = d$. □

In Corollary 5.3.13 we show that slices of μ are homotopically non-trivial on spheres defined in $\mathcal{S}(\eta)^\perp$. By using Proposition 2.1.9, Proposition 2.1.33, and Lemma 5.3.5 we can show this implies the slices of a conical measure μ along it's own spine $\mathcal{S}(\mu)$ are homotopically non-trivial.

Theorem 5.3.15 (The Rigidity Theorem)**.** *For any $\epsilon > 0$ and $\theta > 0$ there is $\delta > 0$ depending on n, N, θ, ϵ such that the following holds. Let $\mu, \eta \in \mathcal{M}$ be conical measures,* *and suppose η is cylindrical with $\Theta_\eta(0) \leq \theta$. Let $\phi, \psi \in W^{1,2}(B_1, N)$ be maps associated to μ, η respectively, and suppose the slices of ψ along $\mathcal{S}(\eta)$ are homotopically non-trivial. Finally suppose*

$$d_{\mathcal{M}}((\mu, \phi), (\eta, \psi)) < \delta, \quad \dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) < d.$$

Then μ is cylindrical, the slices of ϕ along $\mathcal{S}(\mu)$ are homotopically non-trivial, and

$$\mathcal{S}(\mu) \cap B_{\frac{1}{2}} \subset B_\epsilon(\mathcal{S}(\eta)) \cap B_{\frac{1}{2}}. \tag{5.6}$$

Remark 5.3.16. The assumption $\dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) < d$ can be verified if one can show that all singularities $x \in \Sigma(\mu)$ with $\Theta_\mu(x) < \Theta_\mu(0)$ are contained in the $(d-1)$-singular strata $\mathrm{Sing}_{d-1}(\mu)$ of Lemma 4.3.10.

Proof. For sufficiently small $\delta > 0$, by Corollary 5.3.13 we have that

$$\dim_{\mathcal{H}}(\Sigma(\mu)) = d.$$

However by the assumption that $\dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) < d$ we must then have that $\dim(\mathcal{S}(\mu)) = d$, and so μ is cylindrical.

The claim (5.6) follows from Lemma 5.3.3 for sufficiently small $\delta > 0$. It follows from (5.6) and Proposition 2.1.9 that $\mathcal{S}(\mu)$ is a small rotation of $\mathcal{S}(\eta)$, where the size of this rotation is controlled by $\epsilon > 0$.

Since μ is cylindrical, the associated map ϕ is translation-invariant along $\mathcal{S}(\mu)$, and ϕ is invariant under rescales centred on points $x \in \mathcal{S}(\mu)$. As such to show that ϕ is homotopically non-trivial on spheres $C_r(x, \mathcal{S}(\mu))$ for $x \in \mathcal{S}(\mu)$ and $r > 0$, it suffices to prove this at the origin $x = 0$ for any $r > 0$.

At the origin we have by Corollary 5.3.13 that ϕ is homotopically non-trivial on a sphere $C(\eta) = C_{1/2}(0, \mathcal{S}(\eta))$. By Proposition 2.1.9 the sphere $C(\mu) = C_{1/2}(0, \mathcal{S}(\mu))$ is a small rotation of $C(\eta)$, in particular we can define a homotopy between $C(\eta)$ and $C(\mu)$ which lies in $B_1 \setminus B_\epsilon(\mathcal{S}(\mu))$. Since μ is cylindrical, by Lemma 5.3.5, for any $\gamma > 0$ there is $r = r(\gamma, \epsilon, \theta) > 0$ such that $\Theta_\mu(x, s) < \gamma$ for any $x \in B_1 \setminus B_\epsilon(\mathcal{S}(\mu))$ and $s \leq r$. By Bethuel regularity Theorem 4.3.5, for sufficiently small γ this will implies pointwise estimates on $|D\phi|$ at points in $B_1 \setminus B_\epsilon(\mathcal{S}(\mu))$. In particular this implies $\phi\lfloor_{C(\eta)}$ is pointwise close to $\phi\lfloor_{C(\mu)}$, controlled by ϵ. As such by Proposition 2.1.33 it follows that $\phi\lfloor_{C(\eta)}$ is homotopically equivalent to $\phi\lfloor_{C(\mu)}$ for sufficiently small $\epsilon > 0$, which implies ϕ is homotopically non-trivial on $C(\mu)$.

$\square$

As noted in the remark, the condition $\dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) < d$ can be verified when we can show the singularities not in the spine are part of a lower singular stratum. One way to ensure this is to let $\Theta_\mu(0)$ be minimal across the top dimensional stratum. Recall in Definition 4.4.7 the minimum cylindrical density $\alpha \geq \epsilon_0$ is defined.

Lemma 5.3.17. *Suppose $\mu \in \mathcal{M}$ is conical, and $\Theta_\mu(0) = \alpha$. Then*

$$\dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) < d.$$

Remark 5.3.18. In fact we only need α to be minimal among the densities of points in $\mathrm{Sing}_d(\mu)$.

Proof. By the dimension reduction of Lemma 4.3.10 it suffices to show $\Sigma(\mu) \setminus \mathcal{S}(\mu)$ is contained in a lower stratum of the singular set than the top dimensional part. For any $x \in \Sigma(\mu) \setminus \mathcal{S}(\mu)$, since μ is conical we have that $\Theta_\mu(x) < \Theta_\mu(0) = \alpha$, and as such no tangent $\nu \in T_x\mu$ can be cylindrical as $\Theta_\nu(0) = \Theta_\mu(x) < \alpha$. So $\Sigma(\mu)\setminus\mathcal{S}(\mu) \subset \mathrm{Sing}_{d-1}(\mu)$ and as such $\dim_{\mathcal{H}}(\Sigma(\mu) \setminus \mathcal{S}(\mu)) \leq d - 1$. □

Recall that in Corollary 5.2.5 we have that the pseudo-tangents η have $\Theta_\eta(0) = \Theta_\mu(x)$. This means if we assume some fixed μ has $\Theta_\mu(x) = \alpha$, then $\Theta_\eta(0) = \alpha$ for any pseudo-tangent η. As such to apply Theorem 5.3.15 to pseudo-tangents it will suffice that the pseudo-tangents are close together as the base points and scales are changed slightly. As pseudo-tangents are close to rescales $\mu_{y,\lambda}$, it suffices to show that $\mu_{y,\lambda}$ doesn't vary too much as y and λ are varied. In other words $\mu_{y,\lambda}$ are continuous with respect to translation and rescaling. This is the aim of the next section.

5.4 Continuity of Rescaling

In this section we fix a measure $\mu \in \mathcal{M}$ with associated map $u \in W^{1,2}(B_1; N)$, and a singularity $x_0 \in \Sigma(\mu)$. We show that rescaling and translating along $S^+(x_0)$ are continuous operations on μ with respect to the convergence in $\mathcal{M}$.

We split this into two results for simplicity. First we consider purely rescaling. In the following d denotes some metric on $\mathcal{M}$, for example Definition 4.1.9.

Lemma 5.4.1 (Continuity of rescaling)**.** *For each $\epsilon > 0$ there exists $\delta > 0$ depending on μ, x, n, N, ϵ such that the following holds. For any $x \in S^+(x_0) \cap B_\delta(x_0)$ and $r, s < \delta$ such that $\frac{1}{2}r \leq s \leq 2r$, we have that*

$$d_{\mathcal{M}}((\mu_{x,r}, u_{x,r}), (\mu_{x,s}, u_{x,s})) < \epsilon. \tag{5.7}$$

Remark 5.4.2. By letting δ depend on $C > 1$ we can in fact choose $C^{-1}r \leq s \leq Cr$.

Proof. Suppose this were false for some $\epsilon > 0$. Then we can find $x_i \in B_{1/i}(x_0)$ and $r_i, s_i < 1/i$ with $\frac{1}{2}r_i \leq s_i \leq 2r_i$, such that (5.7) fails for each i. By Lemma 5.2.3 we can find a subsequence such that

$$(\mu_{x_i,r_i}, u_{x_i,r_i}) \to (\eta, \phi), \quad (\mu_{x_i,s_i}, u_{x_i,s_i}) \to (\nu, \psi),$$

for conical measures $\eta, \nu \in \mathcal{M}$ and conical maps $\phi, \psi \in W^{1,2}(B_1; N)$. We can choose this subsequence so that $s_i/r_i \to \tau \in [\frac{1}{2}, 2]$. Clearly we have

$$\mu_{x_i,s_i} = (\mu_{x_i,r_i})_{0,\frac{s_i}{r_i}} \to \eta_{0,\tau} = \eta.$$

The final equality follows as η is conical. Since μ_{x_i,s_i} converges to ν we have $\eta = \nu$. Similarly since ϕ is conical we have that

$$u_{x_i,s_i} = (u_{x_i,r_i})_{0,\frac{s_i}{r_i}} \to \phi_{0,\tau} = \phi.$$

This implies $\phi = \psi$ since $u_{x_i,s_i} \to \psi$. As such (5.7) cannot fail along a sequence. $\square$

Next we consider translations along $S^+(x_0)$. Note that we have to restrict to translations along $S^+(x_0)$ as we make use of Lemma 5.2.3.

Lemma 5.4.3 (Continuity of translation)**.** *For any $\epsilon > 0$ there exists $\delta > 0$ depending on μ, x_0, n, N, ϵ such that if $r < \delta$, $x, y \in S^+(x_0) \cap B_r(x_0)$, then we have*

$$d_{\mathcal{M}}((\mu_{x,r}, u_{x,r}), (\mu_{y,r}, u_{y,r})) < \epsilon. \tag{5.8}$$

Proof. Suppose this were false for some $\epsilon > 0$. Then there are $r_i < \frac{1}{i}$, and $x_i, y_i \in S^+(x_0) \cap B_{r_i}(x_0)$ such (5.8) fails for each i. By Lemma 5.2.3 there is a subsequence so that

$$(\mu_{x_i,r_i}, u_{x_i,r_i}) \to (\eta, \phi), \quad (\mu_{y_i,r_i}, u_{y_i,r_i}) \to (\nu, \psi),$$

for conical measures $\eta, \nu \in \mathcal{M}$ and conical maps $\phi, \psi \in W^{1,2}(B_1; N)$. We also have from Lemma 5.2.3 that

$$\Theta_\eta(0) = \Theta_\mu(x_0) = \Theta_\nu(0).$$

Since (5.8) fails for each i it also fails for η, ν in place of $\mu_{x,r}$ and $\mu_{y,r}$, and ϕ, ψ in place of $u_{x,r}, u_{y,r}$. We show for a contradiction that in fact $\eta = \nu$ and $\phi = \psi$.

Let $z_i = \frac{y_i - x_i}{r_i} \in B_2$. Note that since $x_i, y_i \to x_0$ and $r_i \to 0$, for any $R > 0$ the measures μ_{y_i,r_i} and μ_{x_i,r_i} are eventually defined on B_R for sufficiently large i depending on R. Clearly $\mu_{y_i,r_i}(A) = \mu_{x_i,r_i}(A + z_i)$. In particular since $y_i \in S^+(x_0)$ we have that

$$\Theta_{\mu_{x_i,r_i}}(z_i) = \Theta_{\mu_{y_i,r_i}}(0) = \Theta_\mu(y_i) \geq \Theta_\mu(x_0).$$

Now choose a further subsequence such that $z_i \to z \in \overline{B}_2$. Applying upper-semicontinuity of the density we have that

$$\Theta_\eta(z) \geq \limsup_{i\to\infty} \Theta_{\mu_{x_i,r_i}}(z_i) \geq \Theta_\mu(x_0) = \Theta_\eta(0).$$

Therefore $z \in \mathcal{S}(\eta)$ and so by translation invariance along the spine we have $\eta_{z,1} = \eta$. So we have the following

$$\mu_{y_i,r_i} = (\mu_{x_i,r_i})_{z_i,1} \to \eta_{z,1} = \eta.$$

Since the μ_{y_i,r_i} converges to ν we have that $\eta = \nu$. Now note that ϕ is also translation invariant along $\mathcal{S}(\eta)$ by Lemma 4.2.19. So we have that

$$u_{y_i,r_i} = (u_{x_i,r_i})_{z_i,1} \to \phi_{z,1} = \phi.$$

The convergence here is both strong in L^2 and weak in $W^{1,2}$. However u_{y_i,r_i} converges in L^2 and weakly in $W^{1,2}$ to ψ, implying $\phi = \psi$. As such (5.8) can't fail along a subsequence.

$\square$

Combining these two results via the triangle inequality gives the following theorem.

Theorem 5.4.4 (Continuity of rescales on $S^+(x_0)$)**.** *Let $\mu \in \mathcal{M}$ with associated map $u \in W^{1,2}(B_1; N)$, and $x_0 \in \mathrm{Sing}(u)$. For each $\epsilon > 0$ there exists $\delta = \delta > 0$, depending on $\epsilon, x_0, \mu, u, n, N$ such that the following property holds. For any $r, s \in (0, \delta)$ with $\frac{1}{2}r \leq s \leq 2r$ and $x, y \in S^+(x_0) \cap B_s(x_0)$ we have*

$$d_{\mathcal{M}}((\mu_{x,r}, u_{x,r}), (\mu_{y,s}, u_{y,s})) < \epsilon. \tag{5.9}$$

Remark 5.4.5. Note that the assumption $x, y \in B_s(x_0)$ is required rather than $x, y \in B_\delta(x_0)$. We will have to deal with this discrepancy in the next section by an iterative argument.

5.5 Structure Results

The aim of this section is to show a structure result for $\Sigma(\mu) \cap B_r(x)$ given $\mu \in \mathcal{M}$ and a singularity $x \in \Sigma(\mu)$ with the following conditions. There is a cylindrical tangent measure $\eta \in T_x\mu$ with an associated map $\psi \in W^{1,2}(B_1; N)$, and the slices $\psi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ along $\mathcal{S}(\eta)$ are homotopically non-trivial. Further we assume $\Theta_\mu(x) = \Theta_\eta(0) = \alpha$ is the minimal cylindrical density. With these assumptions we show that the pseudo-tangents on $S^+(x) \cap B_r(x)$ are also all cylindrical, and that the spines of these pseudo-tangents satisfy the necessary conditions to apply Reifenberg's approximation Theorem 2.2.3 to $S^+(x) \cap B_r(x)$. Then we use the homotopically non-trivial property to show that all singularities of μ local to x are actually in $S^+(x)$.

Throughout this section $\mu \in \mathcal{M}$ and an associated map $u \in W^{1,2}(B_1; N)$ are fixed. We also fix a singularity $x_0 \in \mathrm{Sing}(\mu)$ at which there is a cylindrical $\eta \in T_x\mu$ with an associated map $\psi \in W^{1,2}(B_1; N)$ such that the slices of ψ along $\mathcal{S}(\eta)$ are homotopically non-trivial.

We will use the rigidity Theorem 5.3.15 to show the pseudo-tangents of Corollary 5.2.5 are cylindrical. Recall the rigidity theorem requires that $\dim(\Sigma(\nu) \setminus \mathcal{S}(\nu)) < d$. One way we can guarantee this is to use the minimal cylindrical density α of Definition 4.4.7. In the following we show that if $\Theta_\mu(x_0) = \alpha$ then the pseudo-tangents satisfy the required dimension condition. This result is similar to Lemma 5.3.17

Lemma 5.5.1 (Pseudo-tangents have top dimensional spines)**.** *Suppose $\mu \in \mathcal{M}$, $x_0 \in \Sigma(\mu)$ and $\Theta_\mu(x_0) = \alpha$. Then any pseudo-tangent $\nu \in \mathcal{M}$ of Corollary 5.2.5 satisfies*

$$\dim_{\mathcal{H}}(\Sigma(\nu) \setminus \mathcal{S}(\nu)) < d.$$

Remark 5.5.2. In fact we only need that α is minimal among the density of any cylindrical tangent to any pseudo-tangent of μ near x_0.

Proof. Recall that $\Theta_\nu(0) = \Theta_\mu(x_0) = \alpha$ by Corollary 5.2.5. As such for any $y \in \Sigma(\nu) \setminus \mathcal{S}(\nu)$ we have $\Theta_\nu(y) < \alpha$, and so no tangent $\nu' \in T_y\nu$ can be cylindrical. Thus by dimension reduction Lemma 4.3.10 we have that $\dim(\Sigma(\nu) \setminus \mathcal{S}(\nu)) < d$ as required. □

In the following we use an iterative argument to show the pseudo-tangents are cylindrical for sufficiently small scales, and points sufficiently close to x_0. We do this by comparing a pseudo-tangents to a measure we know to be cylindrical. We can make this comparison by actually comparing the rescales of μ, making use of Theorem 5.4.4. Then when the pseudo-tangent is sufficiently close to a cylindrical measure with homotopically non-trivial associated map, Theorem 5.3.15 will tell us that the pseudo-tangent is cylindrical, also with a homotopically non-trivial associated map.

Theorem 5.5.3 (Pseudo-Tangents are Cylindrical)**.** *Let $\mu \in \mathcal{M}$, $x_0 \in \Sigma(\mu)$ with $\Theta_\mu(x_0) = \alpha$. Suppose there is $\eta \in T_{x_0}\mu$ which is cylindrical and such that for some $\psi \in W^{1,2}(B_1; N)$ associated to η the slices $\eta_{\mathcal{S}(\eta)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\eta)^\perp$ are homotopically non-trivial.*

For any $\epsilon > 0$ there is $\delta = \delta(\mu, x_0, n, N, \epsilon) > 0$ such that the following holds. If $s \in (0, \delta)$ and $y \in S^+(x_0) \cap B_\delta(x_0)$ then any pseudo-tangent ν at y with scale s is cylindrical, there is a map $\phi \in W^{1,2}(B_1; N)$ associated to ν for which the slices $\phi\lfloor_{\mathcal{S}(\nu)^\perp,0}$ restricted to $\{0\} \times S^{n-d}_{1/2} \subset \mathcal{S}(\nu)^\perp$ are homotopically non-trivial, and

$$\mathcal{S}(\nu) \cap B_{1/2} \subset B_\epsilon(\mathcal{S}(\eta)) \cap B_{1/2}. \tag{5.10}$$

Remark 5.5.4. The minimum cylindrical density assumption can be replaced by any assumption which guarantees $\dim_{\mathcal{H}}(\Sigma(\nu) \setminus \mathcal{S}(\nu)) < d$ for each pseudo-tangent ν.

Proof. Let $\tau > 0$ denote the δ of the rigidity result Theorem 5.3.15 using the ϵ from the current statement. Choose $\hat{\delta} = \hat{\delta}(\epsilon, n, N, x_0, \mu)$ sufficiently small so that Theorem 5.4.4 applies, with ϵ there replaced by $\tau/3$. Further choose $\hat{\delta}$ smaller than the $r = r(\mu, x_0, \tau)$ of Corollary 5.2.5 with ϵ there replaced by $\tau/3$. We can choose a sequence so that $(\mu_{x_0,r_i}, u_{x_0,r_i})$ converge to (η, ψ) for some map $\psi \in W^{1,2}(B_1; N)$ associated to η. Now let δ denote the maximal $r_i < \hat{\delta}/2$ such that

$$d_{\mathcal{M}}((\mu_{x_0,r_i}, u_{x_0,r_i}), (\eta, \psi)) < \frac{\tau}{3}.$$

Now for any $s \in [\delta/2, 2\delta]$ and $y \in S^+_s(x_0)$ let (ν, ϕ) denote the pseudo-tangent of Corollary 5.2.5. By Theorem 5.4.4 we get the following by the triangle inequality

$$\begin{aligned} d_{\mathcal{M}}((\nu, \phi), (\eta, \psi)) &\leq d_{\mathcal{M}}((\nu, \phi), (\mu_{y,s}, u_{y,s})) + d_{\mathcal{M}}((\mu_{y,s}, u_{y,s}), (\mu_{x_0,\delta}, u_{x_0,\delta})) \\ &\quad + d_{\mathcal{M}}((\mu_{x_0,\delta}, u_{x_0,\delta}), (\eta, \psi)) < \tau. \end{aligned}$$

By the choice of τ Theorem 5.3.15 applies, implying ν is cylindrical, and the slices of ϕ along $\mathcal{S}(\eta)$ are homotopically non-trivial. This is true for any pseudo-tangent at scale $s \in [\delta/2, 2\delta]$ and at a point $y \in S^+(x_0) \cap B_s(x_0)$. In particular this applies for any $y \in S^+(x_0) \cap B_\delta(x_0)$ and $s \in [\delta, 2\delta]$. We now want to extend the values s can take down to 0.

We proceed by induction to show that the pseudo-tangents are cylindrical and homotopically non-trivial for $s \in (0, 2\delta]$. Set $\delta_j = 2^{1-j}\delta$ for each $j = 1, 2 \ldots$, and note we can write $(0, 2\delta] = \cup_j [\delta_j/2, 2\delta_j]$. Both the $j = 1$ and inductive case can be proved as follows. For each $j = 1, 2, \ldots$ we know that at any $y \in S^+(x_0)$ and scale δ_j there is a cylindrical homotopically non-trivial pseudo-tangent (ν, ϕ). Then for any pseudo-tangent (ν', ϕ') at y with scale $s \in [\delta_j/2, 2\delta_j]$ we have by Theorem 5.4.4 and the choice of δ earlier that

$$d_{\mathcal{M}}((\eta, \phi), (\eta', \phi')) < \tau.$$

This proves (η', ϕ') are also cylindrical and homotopically non-trivial by the rigidity theorem, Theorem 5.3.15. Finally note that (5.10) follows from Lemma 5.3.3.

□

An interesting corollary is that these properties now extend to the tangents on $S^+(x_0)$ local to x_0.

Corollary 5.5.5 (Properties of tangents near x_0)**.** *Suppose $\mu \in \mathcal{M}$ has an associated map $u \in W^{1,2}(B_1; N)$, and $x_0 \in \Sigma(\mu)$. Further suppose the assumptions of Theorem 5.5.3 hold, and fix $\epsilon > 0$. Then for the $\delta = \delta(\mu, x_0, n, N, \epsilon) > 0$ of Theorem 5.5.3 we have that for any $y \in S^+(x_0) \cap B_\delta(x_0)$, any tangent measure $\eta \in T_y\mu$ is cylindrical with $\Theta_\eta(0) = \alpha$. Any map $\psi \in W^{1,2}(B_1; N)$ that arises as a strong L^2 and weak $W^{1,2}$ limit of u_{y,λ_j} for some null sequence $\lambda_j > 0$ is associated to η. Further the slices $\psi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\eta)^\perp$ are homotopically non-trivial.*

Remark 5.5.6. This is a rigidity result for the tangents at singularities local to x_0 in $S^+(x_0)$.

Proof. Let $\tau > 0$ denote the $\delta > 0$ of rigidity Theorem 5.3.15. By choosing $\delta > 0$ such that Theorem 5.5.3 applies, and Corollary 5.2.5 applies with $\epsilon = \tau/2$ we have that for any $y \in S^+(x) \cap B_\delta(x)$ and $\lambda \in (0, \delta)$, there is a cylindrical pseudo-tangent $\nu_\lambda \in \mathcal{M}$ and associated map $\phi_\lambda \in W^{1,2}(B_1; N)$ such that $\phi_\lambda\lfloor_{\mathcal{S}(\nu)^\perp,0}$ restricted to $\{0\} \times S^{n-d}_{1/2} \subset \mathcal{S}(\nu)^\perp$ is homotopically non-trivial. Further we have that

$$d_{\mathcal{M}}((\mu_{y,\lambda}, u_{y,\lambda}), (\nu_\lambda, \phi_\lambda)) < \tau/2.$$

Then for any $\eta \in T_y\mu$ there is a null sequence $\lambda_j > 0$ such that $\mu_{y,\lambda_j} \to \eta$. Further suppose u_{y,λ_j} converge strongly in L^2 and weakly in $W^{1,2}$ to $\psi \in W^{1,2}(B_1; N)$, which is always possible along some subsequence of λ_j. Then we have that

$$d_{\mathcal{M}}((\eta, \psi), (\nu_{\lambda_j}, \phi_{\lambda_j})) < d_{\mathcal{M}}((\eta, \psi), (\mu_{y,\lambda_j}, u_{y,\lambda_j})) + \tau/2.$$

Since the first term on the right hand side converges to 0, the result follows by the rigidity Theorem 5.3.15. □

Once the pseudo-tangents are cylindrical we can now apply Lemma 5.3.3 to prove one of the necessary inclusions for the Reifenberg approximation Definition 2.2.2. In the following d_* denotes some metrisation of the weak convergence of Radon measures, as in Definition 2.1.24.

Corollary 5.5.7 (Approximation of $\Sigma(\mu)$ by pseudo-tangents)**.** *Under the same assumptions as Theorem 5.5.3 we have the following. For any $\epsilon > 0$ there is $\delta = \delta(\mu, x_0, n, N, \epsilon) > 0$ such that if $s \in (0, \delta), y \in S^+(x_0) \cap B_\delta(x_0)$ and ν is a pseudo-tangent at y with scale s then*

$$\Sigma(\mu) \cap B_s(y) \subset B_{\epsilon s}(\mathcal{S}(\nu) + y) \cap B_s(y).$$

Proof. Let $\tau = \tau(n, M, \epsilon)$ denote the δ of Lemma 5.3.3. Choosing $\delta > 0$ sufficiently small that Lemma 5.2.5 and Theorem 5.5.3 apply with ϵ there replaced by τ, we have that the pseudo-tangents ν are cylindrical, and $d_*(\nu, \mu_{y,s}) < \tau$. Then the result follows from Lemma 5.3.3 since $\Sigma(\mu_{y,s}) = s^{-1}(\Sigma(\mu) - y)$. $\square$

At this point we could apply the David-Toro [DT12] Reifenberg result, stated in Theorem 2.2.5, to $S^+(x_0) \cap B_\delta(x_0)$. The tilt conditions on the planes necessary for this theorem follow from Theorem 5.4.4. Theorem 5.5.3 shows that the planes all have the same dimension. As such $S^+(x_0) \cap B_\delta(x_0)$ would be contained in the bi-Hölder image of a disk.

However the homotopically non-trivial property allows us to show more. For a pseudo-tangent ν at a point $y \in S^+(x_0) \cap B_\delta(x_0)$ at scale $0 < s < \delta$ we can show the following.

$$(y + \mathcal{S}(\nu)) \cap B_s(y) \subset B_{\epsilon s}(S^+(x_0)) \cap B_s(y). \tag{5.11}$$

This can be read as saying $S^+(x_0)$ has no ϵs-gaps in $B_s(y)$. Indeed if we could find a point $y_* \in \mathcal{S}(\eta_{y,s})$ such that $B_{\epsilon s}(y_*)$ doesn't meet $S^+(x_0)$ then we call this an ϵs-gap in $S^+(x_0)$, and this clearly violates (5.11).

Lemma 5.5.8 (No Gaps Lemma)**.** *Fix $\mu \in \mathcal{M}$, $u \in W^{1,2}(B_1; N)$ associated to μ, and consider $x_0 \in \Sigma(\mu)$ with $\Theta_\mu(x_0) = \alpha$ such that there exists a cylindrical tangent $\eta \in T_x\mu$ with associated map $\psi \in W^{1,2}(B_1; N)$. Further suppose $\psi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\eta)^\perp$ is homotopically non-trivial. Then for any $\epsilon > 0$ there is $\delta = \delta(\mu, x_0, n, N, \epsilon) > 0$ such that if $s \in (0, \delta), y \in S^+(x_0) \cap B_\delta(x_0)$ and ν is a pseudo-tangent at y with scale s then*

$$(y + \mathcal{S}(\nu)) \cap B_s(y) \subset B_{\epsilon s}(S^+(x_0)) \cap B_s(y).$$

Proof. If this were false for some $\epsilon > 0$ then for any small $\delta > 0$ we may find $s \in (0, \delta)$, $y \in S^+(x_0) \cap B_\delta(x_0)$ and a pseudo-tangent ν at y with scale s such that there exists $z \in y + \mathcal{S}(\nu) \cap B_s(y)$ with $B_{\epsilon s}(z) \cap S^+(x_0) \cap B_s(y) = \emptyset$. As such we have

$$B_{\epsilon s}(z) \cap \mathrm{Sing}(\mu) \cap B_s(y) \subset \Sigma(\mu) \setminus S^+(x_0).$$

Then by definition of $S^+(x_0)$ we have that

$$\Theta_\mu(z') < \Theta_\mu(x_0) = \alpha, \quad \text{for } z' \in B_{\epsilon s}(z) \cap \Sigma(\mu) \cap B_s(y).$$

This implies no $\eta \in T'_z\mu$ can be cylindrical. Thus by dimension reduction Lemma 4.3.10 we have that $\dim_{\mathcal{H}}(B_{\epsilon s}(z) \cap \Sigma(\mu) \cap B_s(y)) < d$. Now consider the orthogonal projection $P : \mathbb{R}^n \to y + \mathcal{S}(\nu)$. Since $\mathcal{S}(\nu)$ is d-dimensional, but local to z we have that $\Sigma(\mu)$ is at most $(d-1)$-dimensional, it follows that

$$\dim_{\mathcal{H}}(y + \mathcal{S}(\nu) \cap B_{\epsilon s}(z) \cap B_s(y) \setminus P(\Sigma(\mu))) > 0.$$

So we may find $z_* \in \mathcal{S}(\nu) \cap B_{\epsilon s}(z) \setminus P(\Sigma(\mu))$. In fact we can find $z_* \in y + \mathcal{S}(\nu) \cap B_{\epsilon s}(z) \cap B_s(y)$ such that the slice $P^{-1}(z_*) \cap B_{\epsilon s}(z) \cap B_s(y)$ contains no singularities, and is at least $3\epsilon/4$ in length. Note that this wouldn't be possible if for example both z and z_* were very close to $\partial B_s(y)$. The aim is to show this implies u is regular on an $(n-d)$-dimensional sphere centred on z_*, and as such a slice of u along $y + \mathcal{S}(\eta)$ is homotopically trivial.

Define a slice map $\widetilde{u}_{z_*}(\xi) = u(z_* + \xi)$ for $\xi \in \mathcal{S}(\nu)^\perp$. This is regular on $B^{n-d}_{\epsilon s} \subset \mathcal{S}(\phi_{y,s})^\perp$ by choice of z_*, as for any $\xi \in B^{n-d}_{\epsilon s}$ we have that $(z_* + \xi) \in B_{\epsilon s}(z)$. Then $P(z_* + \xi) = z_*$, implying $z_* + \xi$ is not a singularity of μ. As such $\widetilde{u}_{z_*} \lfloor \partial B^{n-d}_{\epsilon s/2}$ is homotopically trivial. Translating and scaling we have the map

$$v(\zeta) = u_{y,s}(z_*/s, \zeta) = \widetilde{u}_{z_*}(s\zeta), \quad \text{for } \zeta \in \partial B^{n-d}_{\epsilon/2}.$$

This map v is also homotopically trivial. Let $\phi \in W^{1,2}(B_1; N)$ denote a map associated to ν. By Lemma 5.2.5 we can choose δ sufficiently small so that

$$d_{\mathcal{M}}((\mu_{y,s}, u_{y,s}), (\nu, \phi)) < \tau$$

where τ denotes the δ from Corollary 5.3.13. Then the slices $\phi\lfloor_{\mathcal{S}(\nu)^\perp, 0}$ on $\partial B^{n-2}_{\epsilon/2} \subset \mathcal{S}(\nu)^\perp$ are homotopically equivalent to v, contrary to the fact that ϕ is homotopically non-trivial by Theorem 5.5.3. $\square$

Before stating the main result we first show all singularities local x_0 are actually in $S^+(x_0)$ by iterating the two inclusions we have.

Lemma 5.5.9 (All singularities are in $S^+(x_0)$)**.** *There is $\delta = \delta(\mu, x_0, n, N) > 0$ such that $S^+(x_0) \cap B_\delta(x_0) = \Sigma(\mu) \cap B_\delta(x_0)$.*

Proof. Fix $\epsilon = 1/4$ in Lemma 5.5.8 and Corollary 5.5.7 and let $z \in \Sigma(\mu) \cap B_\delta(x_0)$ with the corresponding δ from those lemmas. Let ν denote the pseudo tangent at x_0 with scale $s \in (|z|, \delta)$. By Corollary 5.5.7 we have that

$$z \in \Sigma(\mu) \cap B_s(x_0) \subset B_{s/4}(\mathcal{S}(\nu) + x_0) \cap B_s(x_0). \tag{5.12}$$

Now by Lemma 5.5.8 we have that

$$B_{s/4}(\mathcal{S}(\nu) + x_0) \cap B_s(x_0) \subset B_{s/2}(S^+(x_0)) \cap B_s(x_0). \tag{5.13}$$

Combining (5.12) and (5.13) we have that $z \in B_{s/2}(S^+(x_0)) \cap B_s(x_0)$. So we can find $z_1 \in S^+(x_0) \cap B_s(x_0)$ such that $|z - z_1| \leq s/2$. We can now repeat this argument at z_1. Inductively assume there is $z_i \in S^+(x_0) \cap B_s(x_0)$ such that $|z - z_i| < s/2^i$. Then by Corollary 5.5.7 we have that for a pseudo-tangent ν at z_i with scale $s/2^i$ the following holds.

$$z \in \Sigma(\mu) \cap B_{s/2^i}(z_i) \subset B_{s/2^{i+2}}(\mathcal{S}(\nu) + z_i) \cap B_{s/2^i}(z_i).$$

Then by Lemma 5.5.8 we have that

$$B_{s/2^{i+2}}(\mathcal{S}(\nu) + z_i) \cap B_{s/2^i}(z_i) \subset B_{s/2^{i+1}}(S^+(x_0)) \cap B_{s/2^i}(z_i).$$

As such we can find $z_{i+1} \in S^+(x_0) \cap B_s(x_0)$ with $|z - z_{i+1}| \leq s/2^{i+1}$. Since $S^+(x_0)$ is closed this implies $z \in S^+(x_0)$. $\square$

Now let $\epsilon_R > 0$ be the ϵ from the Reifenberg theorem, Theorem 2.2.3. The ϵ_R-Reifenberg approximation of $S^+(x_0) \cap B_\delta(x_0)$ is exactly the inclusions that hold by the conclusions of Lemma 5.5.7 and Lemma 5.5.8. By Lemma 5.5.9 we have that $S^+(x_0) \cap B_\delta(x_0) = \Sigma(\mu) \cap B_\delta(x_0)$. As such we have the following result.

Theorem 5.5.10 (Structure of $\Sigma(\mu)$ near x_0)**.** *Let $\mu \in \mathcal{M}$, $x_0 \in \mathrm{Sing}(\mu)$ with $\Theta_\mu(x_0) = \alpha$, and suppose there is a cylindrical tangent measure $\eta \in T_x(\mu)$. Further suppose there is $\psi \in W^{1,2}(B_1; N)$ associated to η such that the slices $\psi\lfloor_{\mathcal{S}(\eta)^\perp, 0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\eta)^\perp$ are homotopically non-trivial. Then for each $\beta \in (0, 1)$ there is $\delta = \delta(n, N, \mu, x_0, \beta) > 0$ such that $\Sigma(\mu) \cap \overline{B}_\delta(x_0)$ can be mapped onto the d dimensional ball by an invertible $C^{0,\beta}$ map.*

Remark 5.5.11. Note that in this result and the following results we have that $\Sigma(\mu) \cap B_\delta(x_0) = S^+(x_0) \cap B_\delta(x_0)$, and so any $x \in \Sigma(\mu) \cap B_\delta(x_0)$ has a homotopically non-trivial tangent measure with maximal spine dimension. A particular consequence of this is that for any $y_i \in \Sigma(\mu)$ with $y_i \to x_0$, we eventually have that $y_i \in S^+(x_0)$, and so $\Theta_\mu(y_i) = \Theta_\mu(x_0) = \alpha$ and there is a tangent at y_i with d-dimensional spine by Corollary 5.5.5.

In fact we can improve on the assumption that x_0 has minimal cylindrical density. It suffices that at $x_0 \in \Sigma(\mu)$ there is a tangent measure which has maximal spine dimension amongst any conical limit measure to μ, and that $\Theta_\mu(x_0)$ is minimal among the densities of such maps. That is we can replace the maximal spine dimension d and minimal cylindrical density α by the following choices.

$$
\begin{aligned}
d_\mu &= \max\left\{\dim(\mathcal{S}(\eta)) \,:\, \eta \text{ is a conical limit measure to } \mu\right\}, \\
\alpha_\mu &= \inf\left\{\Theta_\eta(0) \,:\, \eta \text{ is a conical limit measure to } \mu \text{ with } \dim(\mathcal{S}(\eta)) = d_\mu\right\}.
\end{aligned}
$$

Note that if μ_{x_i,λ_i} converge to η, where $x_i \to x$ and $\lambda_i \to 0$, and η_{y_i,s_i} converge to ν, where $y_i \to y$ and $s_i \to 0$, then $\mu_{x_i+\lambda_i y_i,\lambda_i s_i}$ converge to ν by a simple diagonal argument. As such the set of limit measures of some fixed $\mu \in \mathcal{M}$ is closed under taking further limit measures.

There are a number of arguments that used the cylindrical property, so these must be checked under these assumptions.

The regularity of cylinders result, Lemma 4.4.2, now assumes we have a limit measure η to μ with $\dim(\mathcal{S}(\eta)) = d_\mu$. We can again show $\mathcal{S}(\eta) = \Sigma(\eta)$ by the same argument, noting that a tangent to η is a limit map of μ.

The compactness of the cylindrical class, Lemma 4.4.4, also only requires the modification that η_i are limit maps to μ, and as such by a diagonal argument so is any subsequential limit of the η_i.

Using these we can then show Lemma 5.3.3 and Theorem 5.3.6 apply, where the cylindrical measure is replaced by a limit measure to some fixed μ with spine dimension d_μ. As such the rigidity theorem, Theorem 5.3.15 also follows.

The existence of pseudo-tangents and continuity of rescale and translation results do not make use of the cylindrical property or the minimal density, and so we can use these without modification to prove the following.

Theorem 5.5.12 (Structure Theorem)**.** *Let $\mu \in \mathcal{M}$, $x_0 \in \operatorname{Sing}(\mu)$ with $\Theta_\mu(x_0) = \alpha_\mu$ and suppose there is $\eta \in T_x\mu$ such that $\dim(\mathcal{S}(\eta)) = d_\mu$, and a map $\psi \in W^{1,2}(B_1; N)$*

associated to η such that the slices $\psi\lfloor_{\mathcal{S}(\eta)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\eta)^\perp$ are homotopically non-trivial. Then for any $\beta \in (0,1)$ there is $\delta = \delta(\mu, x_0, N, n, \beta) > 0$ such that $\Sigma(\mu) \cap \overline{B}_\delta(x_0)$ can be mapped onto the d_μ-dimensional ball by a bi-Hölder continuous map with exponent β.

We can now consider what this says in different cases. The first case is for energy minimising maps.

Corollary 5.5.13 (Structure for Energy Minimising Maps)**.** *Consider an energy minimising map $u \in W^{1,2}(B_1; N)$ and suppose $x_0 \in \mathrm{Sing}(u)$. Let $d_u \leq n-3$ denote the maximal spine dimension of conical limit maps to u. Let α_u denote the minimal density of all conical limit maps of u with spine dimension equal to d_u. Suppose $\Theta_u(x) = \alpha_u$ and there is a tangent map $\phi \in T_x u$ which has $\dim(\mathcal{S}(\phi)) = d_u$. Further suppose that the slices of $\phi\lfloor_{\mathcal{S}(\phi)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\phi)^\perp$ are homotopically non-trivial. Then for any $\beta \in (0,1)$ there is $\delta = \delta(u, x_0, N, n, \beta) > 0$ such that $\mathrm{Sing}(u) \cap \overline{B}_\delta(x_0)$ can be mapped onto a d_u-dimensional ball by a bi-Hölder continuous map with exponent β.*

Proof. This follows from Theorem 5.5.12. Note that due to the compactness theorem for energy minimising maps we do not need to appeal to Radon measures at all. The result follows by setting $\mu = |Du|^2 \mathrm{d}x$, and noting that $d_\mu = d_u$ and $\alpha_\mu = \alpha_u$. □

The next case is for stationary harmonic maps. Here we need the assumption that $d_u \leq n-3$, which cannot be verified in general.

Corollary 5.5.14 (Structure for stationary harmonic maps)**.** *Consider a stationary harmonic map $u \in W^{1,2}(B_1; N)$ and singularity $x_0 \in \mathrm{Sing}(u)$. Suppose that the maximal spine dimension of conical limit measures to u is $d_u \leq n-3$. Let α_u denote the minimal density of all conical limit measures with spine dimension equal to d_u. Suppose $\Theta_u(x) = \alpha_u$ and there is a tangent map $\phi \in T_x u$ with $\dim(\mathcal{S}(\phi)) = d_u$, and the slices of $\phi\lfloor_{\mathcal{S}(\phi)^\perp,0}$ restricted to $\partial D_{1/2}(0) \subset \mathcal{S}(\phi)^\perp$ are homotopically non-trivial. Then for any $\beta \in (0,1)$ there is $\delta = \delta(u, x_0, N, n, \beta) > 0$ such that $\mathrm{Sing}(u) \cap \overline{B}_\delta(x_0)$ can be mapped onto a d_u-dimensional disk by a bi-Hölder continuous map with exponent β.*

Finally we note the following property that occurs as a consequence of Corollary 3.3.23 that states $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap K) = 0$ for any $K \subset\subset B_1$ and stationary harmonic map u.

Corollary 5.5.15 (Topology of weak tangents)**.** *Suppose $u \in W^{1,2}(B_1; N)$ is stationary harmonic, $x_0 \in \mathrm{Sing}(u)$ and there is a tangent measure $\eta \in T_x u$ with* $\dim(\mathcal{S}(\eta)) =$

$n-2$. Further suppose $\Theta_u(x_0) = \alpha_u$ is minimal among all points that have a tangent with spine dimension equal to $n-2$. Then the slices of any map associated to η along $\mathcal{S}(\eta)$ are homotopically trivial.

Remark 5.5.16. In particular this is saying that if $\phi \in W^{1,2}(B_1; N)$ are achieved as the strong L^2 and weak $W^{1,2}$ limit of u_{x,λ_i} for some $\lambda_i \to 0$, then the slices of ϕ along $\mathcal{S}(\eta)$ are homotopically trivial. Here η is a measure arising from a limit of $|Du_{x_i,\lambda_i}|^2 \mathrm{d}x$ with $\dim(\mathcal{S}(\eta)) = n-2$.

Proof. If this were not the case, Theorem 5.5.12 would imply $\mathrm{Sing}(u)$ is the bi-Hölder image of an $(n-2)$-dimensional disk local to x_0. This clearly contradicts the fact that $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap K) = 0$ for any $K \subset\subset B_1$. □

In the particular case that the target $N = S^2$ Brezis-Coron-Lieb [BCL86] have shown that non-constant energy minimising maps $\phi : \mathbb{R}^3 \to S^2$ that are conical are always rotations of the map $\psi : \mathbb{R}^3 \to S^2$ defined by $\psi(x) = x/|x|$. Then for an energy minimising map $u \in W^{1,2}_{loc}(B_1; S^2)$ for $B_1 \subset \mathbb{R}^n$ we know that any tangent map $\phi \in T_x u$ has either $\dim(\mathcal{S}(\phi)) = n-3$, or $\dim(\mathcal{S}(\phi)) < n-3$. In the case $\dim(\mathcal{S}(\phi)) = n-3$ it follows that the slice $\phi\lfloor_{\mathcal{S}(\phi)^\perp,0}$ is an energy minimising map from $\mathbb{R}^3$ to S^2, and so is a rotation of ψ. In particular this implies that there is only one cylindrical density, and any cylindrical tangent map is homotopically non-trivial, as the map ψ has degree 1.

Corollary 5.5.17 (The target $N = S^2$)**.** *Suppose $u \in W^{1,2}_{loc}(B_1; S^2)$ is an energy minimising map. We can find closed subsets $T, S \subset B_1$ such that $\mathrm{Sing}(u) = S \cup T$, and for any $\beta \in (0,1)$ the following properties hold.*

- $\dim_{\mathcal{H}}(T) \leq n-4$
- *For each $x \in S$ there is $r = r(u,x,n,\beta) > 0$ such that*

$$\mathrm{Sing}(u) \cap B_r(x) = S \cap B_r(x). \tag{5.14}$$

- *For each $x \in S$, there is $r = r(u,x,n,\beta) > 0$ such that $S \cap \overline{B}_r(x)$ is the image of an $(n-3)$-dimensional disk under a bi-Hölder map with exponent $\beta > 0$*

Remark 5.5.18. Note that only the scale $r = r(u,x,n,\beta) > 0$ depends on $\beta \in (0,1)$. As such S is locally a bi-Hölder image of a disk, where the Hölder exponent can be any $\beta \in (0,1)$. However we cannot push this up to the Lipschitz case $\beta = 1$.

Proof. This follows by applying Corollary 5.5.13 at any $x \in \mathrm{Sing}_{n-3}(u) \setminus \mathrm{Sing}_{n-4}(u)$, and using the classification of Brezis-Coron-Lieb [BCL86]. Both (5.14) and the fact that T is closed in B_1 follow from Lemma 5.5.9, which implies $\mathrm{Sing}(u) \cap B_r(x) = S \cap B_r(x)$ for sufficiently small $r > 0$ and $x \in S_\beta$. As such for $x_i \in T = \mathrm{Sing}(u) \setminus S$, we cannot have that x_i converge to some $x \in S_\beta$. The fact S is closed follows as it is simply the top dimensional stratum of the singular set. □

In the case $N = S^3$ it was shown by Schoen-Uhlenbeck [SU84] that there are no non-constant energy minimising tangent maps $\phi : \mathbb{R}^3 \to S^3$. As such for an energy minimising map $u \in W^{1,2}(B_1; S^3)$ we have that $d_u \leq n-4$. In particular the slices of tangent maps to u are conical energy-minimising maps from $\mathbb{R}^4$ to S^3. Nakajima [Nak06] shows that such maps are homogeneous extensions of an isometry on S^3, and have degree ± 1. In particular if $u \in W^{1,2}_{loc}(B_1; S^3)$ is energy-minimising and $x \in \mathrm{Sing}_{n-4}(u) \setminus \mathrm{Sing}_{n-5}(u)$, then any $\phi \in T_x u$ has homotopically non-trivial slices along $\mathcal{S}(\phi)$. There is a unique cylindrical density in this case also due to the classification of tangent maps as homogeneous extensions of isometries of S^3.

Corollary 5.5.19 (The target $N = S^3$)**.** *Suppose $u \in W^{1,2}_{loc}(B_1; S^3)$ is an energy minimising map. We can find closed subsets $T, S \subset B_1$ such that $\mathrm{Sing}(u) = S \cup T$, and for any $\beta \in (0,1)$ the following properties hold. following properties.*

- $\dim_{\mathcal{H}}(T) \leq n-5$
- *For each $x \in S$ there is $r = r(u, x, n, \beta) > 0$ such that*

$$\mathrm{Sing}(u) \cap B_r(x) = S \cap B_r(x). \tag{5.15}$$

- *For each $x \in S$, there is $r = r(u, x, n, \beta) > 0$ such that $S \cap \overline{B}_r(x)$ is the image of an $(n-4)$-dimensional disk under a bi-Hölder map with exponent $\beta > 0$*

Remark 5.5.20. As in Remark 5.5.18, only $r = r(u, x, n, \beta) > 0$ depends on β, however the proof does not show that S is a bi-Lipschitz image.

Proof. This follows by applying Corollary 5.5.13 at any $x \in \mathrm{Sing}_{n-4}(u) \setminus \mathrm{Sing}_{n-5}(u)$, and using the classification of Nakajima [Nak06]. The fact that T and S are closed subsets of B_1, and the statement (5.15) follow by similar arguments to Corollary 5.5.17. □

Chapter 6

Introduction - Mean Curvature Flows

6.1 Background

Given an n-dimensional surface $M_0 \subset \mathbb{R}^{n+1}$ the mean curvature flow is the evolution $\{M_t\}_{t\geq 0}$ of M_0 by it's mean curvature vector. A trivial example is a hyperplane $M_0 = \mathbb{R}^n \times \{0\}$ that simply remains stationary $M_t = M_0$ for all $t \geq 0$. Indeed any minimal surface is a stationary solution to the mean curvature flow.

A collection of examples are given by the shrinking cylinders $\mathbb{R}^k \times S^{n-k}_{r_0}$ for $k = 0, \ldots, n-1$, where S^{n-k} is the $(n-k)$-dimensional sphere in $\mathbb{R}^{n-k+1}$. In the $k = 0$ case we treat this as the sphere $S^n_{r_0}$. These evolve by self similarly shrinking $M_t = \mathbb{R}^k \times S^{n-k}_{r_t}$, where $r_t = \sqrt{r_0^2 - 2(n-k)t}$. At time $t = \frac{r_0^2}{2(n-k)}$ the radius is $r_t = 0$, and the cylinder collapses on its rotational axis at this time. This demonstrates that an initially smooth surface may form singularities in finite time under mean curvature flow, and further that this singular set may be up to $(n-1)$-dimensional for the flow of an n-dimensional surface. In fact we can consider the flow and its singular set as subsets of space time, in which case the cylinder $\mathbb{R}^{n-1} \times S^1$ collapses on a parabolic Hausdorff dimension $(n-1)$-singular set for a single time. In the case of a sphere, an initially smooth and bounded surface forms a singularity in finite time. By the maximum principle for parabolic partial differential equations, since any smooth bounded initial surface M_0 is initially contained in a large sphere S^n_R, the mean curvature flow of M_0 must form a singularity by time $t = \frac{R^2}{2n}$. See Proposition 1.4 of Ecker's book [Eck04] for more details.

In the above cases the singularities all form at the final time. This is always the case for embedded planar curves as shown by Grayson [Gra87], and in fact the curve always shrinks to a round point as shown by Gage-Hamilton [GH86], meaning that an appropriate rescale of the curve just before the singularity forms is asymptotically close to a circle. This behaviour is not true of embedded n-dimensional surfaces for $n \geq 2$. For example one can construct a dumbbell surface that forms a neck-pinch singularity before vanishing. One way to construct this is to take the self-shrinking torus of Angenent [Ang92] and construct a surface of two large spheres connected via a thin cylinder that threads through the torus. If the spheres are chosen with a sufficiently large radius such that the Angenent torus shrinks to a point before the spheres have disappeared, the maximum principle forces the dumbbell surface to form a neck pinching singularity. In particular the formation of a singularity has allowed the topology of the initial surface to change. This shows that understanding the behaviour of singularities is fundamental to understanding mean curvature flow.

The singularities so far seem to only form a parabolic Hausdorff dimension $(n-1)$-set. Recall that in parabolic Hausdorff dimension, the time dimension counts as two dimensions. A trivial example of a singular set with parabolic dimension $(n+2)$ is a stationary density two hyperplane. Another example where the parabolic dimension of the singular set is $(n+1)$ is the stationary triple junction formed by three n-dimensional half-planes meeting in equal angles along an $(n-1)$-dimensional subspace. The points on the $(n-1)$-dimensional subspace are all singularities, and since one evolution of this surface is to remain stationary at all times, the spacetime singular set is parabolic $(n+1)$-dimensional.

In the above we occasionally refer to "the" mean curvature flow starting at some surface. However once singularities form we must extend our definition of a mean curvature flow to a wider range of surfaces. One such way is to use geometric measure theory, in particular varifolds, according to the definition of Brakke flows [Bra78]. In this case there is no longer uniqueness of the flow, in fact Brakke's definition allows for arbitrary non-uniqueness by simply letting the flow disappear instantaneously at any time.

6.2 Singularities

The main approach to studying singularities is to consider parabolic rescales of the space-time track of the flow centred on the singularity. In the limit as we take larger and larger rescales, the resulting limiting flow is called a tangent flow. The existence of tangent flows relies on working in a class of flows that has some compactness properties. One such useful class are Brakke flows [Bra78], which are evolutions of geometric

Radon measures known as varifolds. A key tool both for proving the existence of tangent flows and analysing singularities is the Huisken monotonicity formula [Hui90] which states that mass of the surfaces in the flow, weighted by a Gaussian heat kernel, is monotonic as we take smaller and smaller scales for the Gaussian around some point. One result of this is that tangent flows evolve by shrinking self-similarly for negative times. This monotonicity and the resulting Gaussian density share many properties with the monotonicity formula and density of minimal surfaces, and the energy density of stationary harmonic maps. In general there is no uniqueness of tangent flows, two sequences of parabolic rescales about some point may lead to two different tangent flows. However in the case that there is a shrinking cylinder as a singularity at some point $X = (x, t)$ it has been shown by Colding-Ilmanen-Minicozzi [CIM15] that any other tangent at X is a shrinking cylinder, and further by Colding-Minicozzi [CM15] that this has the same axis. In other words, cylindrical tangents are unique. We will make use of both of these results to study the structure of the singular set local to a singularity that has a cylindrical tangent.

It was shown by White [Whi97] that the singular set of a Brakke flow can be stratified according to the parabolic spine dimension of the tangent flows at a singularity. The spine of a tangent flow $\{\mathcal{N}_t\}_{t<0}$ is the subspace of translations under which $\mathcal{N}_t$ is invariant. For example the shrinking cylinder $\mathbb{R}^k \times S^{n-k}$ has a k-dimensional spine. If the tangent flow is invariant under translations in time, including for non-negative times, this counts for 2 dimensions in the parabolic spine dimension. As such we can define the parabolic spine dimension as the dimension of spatial translations that leave $\mathcal{N}_{-1}$ invariant, plus 2 if $\mathcal{N}_t$ is a stationary cone for all $t \in \mathbb{R}$. The k-th strata of the singular set is then the set of singularities such that any tangent flow has at most parabolic k-dimensional spine. White shows this k-th strata has parabolic dimension at most k. As such if we are interested in studying a parabolic $(n-1)$-dimensional part of the singular set, it is most important to consider the $(n+1), n$ and $(n-1)$ strata, as the $(n-2)$ and lower strata are lower dimensional.

6.3 Structure of the Singular Set

A structure result for the singular set of a mean curvature flow was shown by Colding-Minicozzi [CM16] in the case that the mean curvature flow has only cylindrical singularities, and is initially a smooth closed embedded surface. It was shown that the singular set is contained in a finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds, and an $(n-2)$-dimensional set.

In fact the method here can be applied to any compact subset of the singular set on which the singularities all admit shrinking cylinders as tangents. The main result here is to identify a natural subset $S^+(X)$ where this occurs local to a cylindrical singularity X. The main result is Theorem 8.4.5, stated as follows.

Theorem (The Structure Theorem for Brakke Flow)**.** *Let $\mathcal{M}$ denote an n-dimensional integral Brakke flow for times $t \in [0,T)$ and suppose $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$. Further suppose that some tangent flow at X_0 is a rotation of the shrinking cylinder flow $\mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ for some $k \geq 0$. Let $P_\delta(X_0) = B_\delta(x_0) \times (t_0 - \delta^2, t_0 + \delta^2)$ denote the parabolic ball of radius $\delta > 0$. Let $S^+(X_0) \subset \mathrm{Sing}(\mathcal{M})$ denote the collection of $Y \in \mathrm{Sing}(\mathcal{M})$ with Gaussian density $\Theta_{\mathcal{M}}(Y) \geq \Theta_{\mathcal{M}}(X)$. Then for some $\delta = \delta(n, k, \mathcal{M}, X_0) > 0$ we have that $S^+(X_0) \cap \overline{P}_\delta(X_0)$ is contained in a finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds and a parabolic $(n-2)$-dimensional set.*

Further we identify a class of flows, namely those arising from Ilmanen's [Ilm94] elliptic regularisation procedure, such that the structure result holds on the whole singular set local to a cylindrical singularity X with spine dimension $(n-1)$, not just the subset $S^+(X)$. This follows by ruling out certain low density tangents near to X with spine dimension at least n. In the following a unit regular flow is $\mathcal{M}$ such that $\Theta_{\mathcal{M}}(X) = 1$ implies $X \in \mathrm{Reg}(\mathcal{M})$.

Corollary (The Structure Theorem for flows arising from elliptic regularisation)**.** *Let $\mathcal{M}$ denote a unit regular n-dimensional Brakke flow in $\mathbb{R}^{n+1}$, for times $t \in [0,T)$, arising from elliptic regularisation. Suppose $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$, and further suppose there is a tangent flow to $\mathcal{M}$ at X_0 which is a rotation of the shrinking cylinder flow $\mathbb{R}^{n-1} \times S^1_{\sqrt{-2t}}$. Then there is $\delta = \delta(\mathcal{M}, X_0, n) > 0$ such that $\mathrm{Sing}(\mathcal{M}) \cap P_\delta(X_0)$ is contained in the union of a finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds of $\mathbb{R}^{n+1}$, and a set with parabolic Hausdorff dimension at most $(n-2)$.*

A particular singularity we have to rule out are the triple junction singularities, points at which a tangent flow is a static or quasi-static union of 3 hyperplanes meeting along an $(n-1)$-dimensional axis. In the case of flows arising from elliptic regularisation we use a parity result to rule out such triple junctions. Triple junctions of the network flow have been studied by Tonegawa-Wickramasekera [TW16] where it is shown that weak closeness to a triple junction implies smooth closeness. Schulze-White [SW17] study a class of flows that can form triple junctions, but not higher density junctions, and show that in weak closeness to a triple junction also implies smooth closeness in this class.

6.4 Outline of Sections

In Chapter 7 we will review the definitions of Brakke flow, and the relevant standard results on monotonicity, Gaussian density, tangents, regularity, and stratification. The results of Colding-Ilmanen-Minicozzi [CIM15] and Colding-Minicozzi [CM15] for cylindrical tangent flows are stated in section 7.4.

In Chapter 8 we present the argument to prove that all singularities in $S^+(X)$ are cylindrical when X is a cylindrical singularity. This follows similar steps to the argument for stationary harmonic maps. However we are able to make use of the rigidity result of Colding-Ilmanen-Minicozzi [CIM15]. The steps of this argument are outlined below.

In section 8.2 we define pseudo-tangent flows on a subset $S^+(X)$ of the singular set in a similar way as to stationary harmonic maps, by making use of limit flows and metrising the convergence of Brakke flows. In particular these pseudo-tangent flows are self-similarly shrinking, and well approximate the parabolic rescales of the original flow about points in $S^+(X)$.

The continuity of the pseudo-tangents under translation along $S^+(X)$ and parabolic dilation is shown in section 8.3.

By a rigidity result for shrinking cylinders of Colding-Ilmanen-Minicozzi [CIM15] and an iteration argument, we show in section 8.4 that there is a fixed radius about a cylindrical singularity on which the pseudo-tangents are all shrinking cylinders. As this scale is fixed, but the scale of the pseudo-tangents is free to vary, we can use this to show there is a cylindrical tangent flow at all points in $S^+(X)$. At this point the arguments of Colding-Minicozzi [CM16] can be applied to $S^+(X)$ on compact subsets.

Finally in section 8.5 we give an argument to show that for unit regular flows with an even-parity property the set $S^+(X)$ is locally the top dimensional part of $\mathrm{Sing}(\mathcal{M})$ for a cylindrical singularity $X \in \mathcal{M}$ with $(n-1)$-dimensional spine. The unit regular flows arising from Ilmanen's elliptic regularization procedure are a subclass of such flows.

Chapter 7

Mean Curvature flows and Brakke flows

7.1 Background

In this section we briefly introduce mean curvature flows and then study Brakke flows. Given a smooth n-dimensional closed hypersurface $M_0 \subset \mathbb{R}^{n+1}$ with inward pointing normal vector $\mathbf{n} : M_0 \to \mathbb{R}^{n+1}$ we can define the mean curvature by

$$H = \mathrm{div}_{M_0} n.$$

Here div_{M_0} is the divergence taken according to the tangent spaces of M_0. The mean curvature vector is $H\mathbf{n}$. The mean curvature flow is the evolution of M_0 with velocity equal to $H\mathbf{n}$. Formally this means that if $F_t : M_0 \to \mathbb{R}^{n+1}$ are a family of embeddings with $F_0(x) = x$ and $M_t = F_t(M_0)$, then

$$\frac{\partial F_t}{\partial t}(x) = H_{M_t}(F_t(x))\mathbf{n}_{M_t}(F_t(x)). \tag{7.1}$$

Here H_{M_t} and $\mathbf{n}_{M_t}$ denote the mean curvature and normal vectors of $M_t = F_t(M_0)$. Since the mean curvature vector can also be computed as the Laplace-Beltrami operator Δ_{M_t} applied to $F_t(x)$, the partial differential equation (7.1) can be seen as a geometric heat equation. It is non-linear since $H_{M_t}, \mathbf{n}_{M_t}$ and Δ_{M_t} all depend on M_t.

Clearly we cannot use smooth embedded surfaces once singularities arise. It turns out that a useful way to study the mean curvature flow past singularities is to replace the surfaces by varifolds, a measure theoretical extension of a surface. Indeed varifolds had been used to prove regularity theorems for minimal surfaces by Allard [All72], and the mean curvature flow analogue of this regularity result is due to Brakke [Bra78].

One key property of both the extension of minimal surfaces to stationary varifolds, and mean curvature flows to Brakke flows, are compactness theorems with respect to weak convergence of measures.

Below we introduce varifolds and the first variation of a varifold. Let $G_n(n+k)$ denote the space of n-dimensional subspaces of $\mathbb{R}^{n+k}$. For a thorough background on varifolds see Simon's notes [Sim83b]. Recall a Radon measure μ on a metric space X is a measure such that $\mu(K) < \infty$ for any $K \subset\subset X$. By Riesz representation we can define a Radon measure by its action on compactly supported continuous functions $\phi \in C_c(X)$ according to

$$\mu(\phi) = \int_X \phi \, \mathrm{d}\mu.$$

Definition 7.1.1 (Varifold)**.** A n-varifold on an open set $U \subset \mathbb{R}^{n+k}$ is a Radon measure on $U \times G_n(n+k)$. The weight measure $\|V\|$ is the Radon measure on U defined by

$$\|V\|(\phi) = \int_{U \times G_n(n+k)} \phi \, \mathrm{d}V, \quad \phi \in C_c(U).$$

The first variation of a varifold is the functional acquired by taking the first variation of the integral of a vector field with respect to variations of the varifold. To understand the first variation we need to understand first what it means to take the variation of a varifold. The following definition can be found in section 39 of Simon's notes [Sim83b].

Definition 7.1.2 (Variation of Varifolds)**.** Let V be an n-varifold on an open subset $U \subset \mathbb{R}^{n+k}$ and consider a mapping $f \in C^1(U; \mathbb{R}^{n+k})$, and further suppose f is proper, it maps compact sets to compact sets. Let $\mathrm{d}f_x : G_n(n+k) \to G_n(n+k)$ denote the linear functional associated to the derivative of f at $x \in U$, restricted to acting on $G_n(n+k)$. Then for $S \in G_n(n+k)$ the action of $\mathrm{d}f_x$ on the basis vectors of S can be represented by a matrix $D_x(S) = \mathrm{d}f_x \lfloor S$. Define the Jacobian by

$$J_S f(x) = \sqrt{\det\left(D_x(S)^T D_x(S)\right)}.$$

Let $G_n^+(U) = \{(x, S) \in G_n(U) \, : \, J_S f(x) \neq 0\}$ and define $F : G_n^+(U) \to G_n(f(U))$ by the map $F(x, S) = (f(x), \mathrm{d}f_x(S))$. Then the variation of V by f is defined by the varifold $f_\# V$ where

$$f_\# V(A) = \int_{F^{-1}(A)} J_S f(x) \, \mathrm{d}V(x, S), \quad A \subset G_n(f(U)), \ A \text{ is Borel}.$$

Remark 7.1.3. The varifold $f_{\#}V$ is called the push-forward of V by f.

The first variation is then defined by the derivative of the mass measures of a family of varifolds that arise from a taking a variation of a varifold under a family of functions.

Definition 7.1.4 (First Variation)**.** Let V be an n-varifold on open subset $U \subset \mathbb{R}^{n+k}$, and let $\{\phi_t\}_{t\in(-1,1)}$ be a family of $C^1(U;\mathbb{R}^{n+k})$ maps, such that $\phi_0(x) = x$. Further suppose $\phi : (-1,1) \times U \to \mathbb{R}^{n+k}$ defined by $\phi(t,x) = \phi_t(x)$ are C^1 in the first variable also. Finally assume the variations are compactly supported in the sense that there is a compact set $K \subset\subset U$ such that $\phi_t(x) = x$ for all $x \in U \setminus K$. Let $X \in C^1_c(U, \mathbb{R}^{n+k})$ denote the vector field achieved by taking the derivative $\frac{\partial \phi_t}{\partial t}\big|_{t=0}$.

The first variation $\delta V : C^1_c(U;\mathbb{R}^{n+k}) \to \mathbb{R}$ is defined by

$$\delta V(X) = \frac{\mathrm{d}}{\mathrm{d}t} \|(\phi_t)_{\#} V \llcorner G_n(K))\| \bigg|_{t=0}.$$

Remark 7.1.5. It should be clear that any $X \in C^1_c(U;\mathbb{R}^{n+k})$ can be represented as the velocity of some variation as described above. Further this definition of first variation is independent of the variation up to its velocity X.

The geometric picture is that V represents a surface and ϕ_t some smooth local variation of the surface. The first variation measures how the area of the surface changes according to this variation. In particular if the surface is locally area minimizing, all variations will increase the area, and so the first variation is zero.

The above definition is clear in what the first variation represents geometrically, but not very useful computationally. By the computations given in section 9 of Simon's notes [Sim83b] we can get the following useful formula.

Proposition 7.1.6 (Divergence Formula for First Variation)**.** *Let V be an n-varifold on an open subset $U \subset \mathbb{R}^{n+k}$, and $X \in C^1_c(U;\mathbb{R}^{n+k})$.*

Given $S \in G_n(n+k)$ let $\tau_1, \dots, \tau_n$ denote an orthonormal basis on S. Define the directional derivatives

$$D_{\tau_i} X(x) = \frac{\partial}{\partial t} X(x + t\tau_i)\bigg|_{t=0}.$$

Then the divergence on S is defined by

$$\operatorname{div}_S(X) = \sum_{i=1}^{n} \tau_i \cdot D_{\tau_i} X.$$

Given these definitions, the first variation satisfies the following.

$$\delta V(X) = \int \mathrm{div}_S X \, \mathrm{d}V(x, S).$$

General varifolds do not have much geometric structure. We restrict to a subclass of integral varifolds for which the support of the weight measure is a countably rectifiable set with integer density. Recall $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$.

Definition 7.1.7 (Countably Rectifiable Set)**.** We say $S \subset \mathbb{R}^{n+k}$ is countably n-rectifiable if

$$S = S_0 \cup \bigcup_{i=1}^{\infty} F_i(A_i)$$

where $\mathcal{H}^n(S_0) = 0$, and $F_i : A_i \to \mathbb{R}^{n+k}$ are Lipschitz.

Remark 7.1.8. By an extension theorem for Lipschitz functions, it is equivalent that S is contained in a countable union of Lipschitz images of $\mathbb{R}^n$, and an $\mathcal{H}^n$-null set. Further by approximation it is also equivalent that the countable union of Lipschitz images is replaced by a countable union of C^1 submanifolds.

Countably rectifiable sets admit approximate tangent spaces with density at $\mathcal{H}^n$-a.e. point. For details of these results see section 11 of Simon's notes [Sim83b].

Integral varifolds are defined by a countably rectifiable set and a non-negative integer valued density function.

Definition 7.1.9 (Integral Varifold)**.** An n-varifold V on open subset $U \subset \mathbb{R}^{n+k}$ is integral if there is a countably n-rectifiable set $M \subset U$ and a locally $\mathcal{H}^n$-integrable function $\theta : M \to \mathbb{N}_0$ such that for any $\phi \in C_c(G_n(U))$ we have

$$V(\phi) = \int_M \phi(x, T_x M)\theta(x) \, \mathrm{d}\mathcal{H}^n(x).$$

If $\theta = 1$ for $\mathcal{H}^n$-a.e. $x \in M$ we say V is a unit density integral varifold.

Remark 7.1.10. In the above the following notation is used $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, \ldots\}$.

In some circumstances we can represent $\delta V(X)$ as an integral of $X \cdot h$ with respect to $\|V\|$. One condition for this is that the total variation is absolutely continuous with respect to $\|V\|$. We define total variation of δV as follows.

Definition 7.1.11 (Total Variation of δV)**.** Let V be an n-varifold on an open subset $U \subset \mathbb{R}^{n+k}$. For any subset $W \subset U$ define

$$\|\delta V\|(W) = \sup\left\{\delta V(X) \,:\, X \in C^1_c(U;\mathbb{R}^{n+k}), \|X\|_{L^\infty(U)} \leq 1, X = 0 \text{ on } U \setminus W\right\}.$$

If $\|\delta V\|$ is absolutely continuous with respect to the weight measure $\|V\|$ we can use the Reisz representation theorem to show the existence of a $\|V\|$-measurable vector field h such that

$$\delta V(X) = -\int_U X \cdot h \,\mathrm{d}\|V\|, \quad X \in C_c(U;\mathbb{R}^{n+k}). \tag{7.2}$$

Definition 7.1.12 (Generalized Mean Curvature Vector)**.** Let V denote an n-varifold on an open subset $U \subset \mathbb{R}^{n+k}$ and suppose $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Then we define the generalised mean curvature vector by h such that (7.2) holds.

With these definitions we can define Brakke flows. These flows were originally studied by Brakke [Bra78] to prove a regularity theorem. This method has since been updated by Kasai-Tonegawa [KT14] to make use of Huisken's monotonicity formula [Hui90], and many of the definitions here can be found in their paper.

Definition 7.1.13 (Brakke Flow)**.** An n-dimensional Brakke flow on an open subset $U \subset \mathbb{R}^{n+k}$, for times $t \in [a,b]$, is a 1-parameter family of Radon measures $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ such that the following holds.

For $\mathcal{L}^1$-almost every $t \in [a,b]$ we have that $\mu_t = \|V_t\|$ for an integral n-varifold V_t on U. Further for any $\phi \in C^1_c(U \times [a,b])$ with $\phi \geq 0$, and any $a \leq t_1 \leq t_2 \leq b$ we have that

$$\int \phi(\cdot,t)\,\mathrm{d}\mu_t\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2}\int -|h|^2\phi + h\cdot\nabla\phi + \frac{\partial\phi}{\partial t}\,\mathrm{d}\mu_t\mathrm{d}t. \tag{7.3}$$

Remark 7.1.14. In the case μ_t are the mass measures associated to the smooth mean curvature flow of surfaces, the inequality (7.3) is an equality. The inequality allows for mass of a Brakke flow to suddenly disappear, however we need to allow for this as only the inequality is preserved under weak limits of Radon measures.

Note that in (7.3) it is implicit that for any Brakke flow the weak mean curvature h exists for almost every time, and is bounded in L^2.

In the following we will always work with the case $k = 1$, the codimension 1 case. This isn't necessary for all of the background material, but will be important later.

A useful result following from (7.3) is that Brakke flows have a uniform mass bound.

Proposition 7.1.15 (Uniform Mass Bounds)**.** *Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ denote an n-dimensional integral Brakke flow on an open subset $U \subset \mathbb{R}^{n+1}$. Then for each $K \subset\subset U$ there is $c_K > 0$ such that $\mu_t(K) \le c_K < \infty$ for all $t \in [a,b]$.*

Proof. This follows from the fact that μ_a is a Radon measure, and the fact that $\mu_t(\phi^4)$ is decreasing in t for the test function $\phi(x,t) = \max(r^2 - |x|^2 - 2nt, 0)$, for any $r > 0$. The monotonicity of $\mu_t(\phi^4)$ follows purely from computation. □

The mass measures μ_t can be discontinuous in time in the sense that for some fixed $\phi \in C^1_c(U \times [a,b])$ we may have that $\mu_t(\phi)$ is discontinuous at some time. However this is only an issue on a null set of times.

Proposition 7.1.16 (Continuity of mass)**.** *Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ be a Brakke flow on an open set $U \subset \mathbb{R}^{n+1}$. Then there is an $\mathcal{L}^1$-null set $Z \subset [a,b]$ such that for any $\phi \in C^1_c(U)$ with $\phi \ge 0$ the function $t \to \mu_t(\phi)$ is continuous for all $t \in [a,b] \setminus Z$.*

Remark 7.1.17. Note that ϕ here is independent of time. In fact we could simply extend $\hat{\phi}(x,t) = \phi(x)$.

Proof. By approximation it suffices to find a null set $Z_i \subset [a,b]$ associated to a countable collection $\phi_i \in C^2_c(U)$ which is dense in $C^1_c(U)$, as then $Z = \cup_i Z_i$.

As such it suffices to show almost everywhere continuity for $t \to \mu_t(\phi)$ for some arbitrary fixed $\phi \in C^2_c(U)$. To this end we note that for any $a \le t_1 < t_2 \le b$ we can apply Cauchy-Schwarz to (7.3) to acquire the following.

$$\mu_{t_2}(\phi) - \mu_{t_1}(\phi) \le \int_{t_1}^{t_2} \frac{1}{2}\frac{|D\phi|^2}{\phi}\,\mathrm{d}\mu_t\mathrm{d}t.$$

The function $\frac{|D\phi|^2}{\phi}$ is bounded by a constant $C(\phi) > 0$ for any $\phi \in C^2_c(U)$, in particular it cannot become unbounded. As such the integral on the right hand side above is bounded as follows

$$\int_{t_1}^{t_2} \frac{1}{2}\frac{|D\phi|^2}{\phi}\,\mathrm{d}\mu_t\mathrm{d}t \le (t_2 - t_1)C(\phi, \mu_a).$$

Note that here we make use of the uniform bounds on mass of Proposition 7.1.15. Since $t_1 < t_2$ was arbitrary we have that the following function is decreasing.

$$t \to \mu_t(\phi) - C(\phi, \mathrm{spt}(\phi), \mathcal{M})t.$$

Since this function is monotonic it is continuous almost everywhere, which by virtue of the continuity of $C(\phi, \mathrm{spt}(\phi), \mathcal{M})t$ implies $t \to \mu_t(\phi)$ is continuous almost everywhere. □

The convergence of Brakke flows is the weak convergence of Radon measures at almost every time.

Definition 7.1.18 (Convergence of Brakke Flows)**.** Let $\mathcal{M}_i = \{\mu_t^i\}_{t\in[a,b]}$ denote a sequence of Brakke flows and $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ another Brakke flow. We say $\mathcal{M}_i \to \mathcal{M}$ converges if $\mu_t^i \to \mu_t$ weakly as Radon measures for $\mathcal{L}^1$-a.e. $t \in [a, b]$.

Remark 7.1.19. The compactness theorem in fact gives a little more than this, including varifold convergence at almost every time along a further subsequence, though this further subsequence may depend on the time. However we will not make use of these additional convergence properties.

The following is a simple but useful result about convergence of Brakke flows.

Proposition 7.1.20 (Convergence of Mass)**.** *Suppose $\mathcal{M}_i = \{\mu_t^i\}_{t\in[a,b]}$ are Brakke flows converging to a Brakke flow $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ on some open subset $U \subset \mathbb{R}^{n+k}$. Then for any $x \in U$, almost every $r > 0$ and almost every t we have that $\mu_t^i(B_r(x)) \to \mu_t(B_r(x))$.*

Proof. This follows immediately from the definition of convergence of Brakke flows and Proposition 2.1.17. □

It will be useful to metrise the convergence of Brakke flows so that later we can quantify two Brakke flows being close to each other. To metrise the convergence of Brakke flows we must account for the convergence $\mu_t^i \to \mu_t$ for $\mathcal{L}^1$-almost every t.

Proposition 7.1.21 (Metrisation of Brakke Convergence)**.** *Fix $a < b$, a compact set $K \subset \mathbb{R}^{n+1}$, and $\Lambda > 0$. Consider the set of Brakke flows $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ on $K \times [a, b]$ with $\mu_t(K) \le \Lambda$ for all $t \in [a, b]$. The weak convergence of such Brakke flows is metrisable.*

Proof. Let $\phi_j \in C_c^0(K)$ be dense in the set of $\phi \in C_c^0(K)$ with $|\phi| \le 1$, and suppose $|\phi_j| \le 1$. Then given two Brakke flows $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}, \mathcal{N} = \{\nu_t\}_{t\in[a,b]}$ on $K \times [a, b]$ with $\mu_t(K) \le \Lambda, \nu_t(K) \le \Lambda$ for all $t \in [a, b]$, we define

$$d(\mathcal{M},\mathcal{N}) = \sum_{j=1}^{\infty} 2^{-j} \int_a^b |\mu_t(\phi_j) - \nu_t(\phi_j)| \, \mathrm{d}t.$$

Since $\mathrm{spt}(\phi_j) \subset K$ and both $\mu_t(K), \nu_t(K)$ are bounded by Λ we have that for any $\epsilon > 0$ there is $J = J(\epsilon, \Lambda, b-a) > 0$ so that the following holds.

$$\sum_{j=J}^{\infty} 2^{-j}(b-a)\,|\mu_t(\phi_j) - \nu_t(\phi_j)| \leq 2^{-J+2}(b-a)\Lambda < \epsilon. \tag{7.4}$$

Now suppose $\mathcal{M}_i = \{\mu_t^i\}_{t\in[a,b]}$ converge as Brakke flows to $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$. Then for any $\epsilon > 0$ we can choose $J > 0$ sufficiently large so that (7.4) holds. By definition of Brakke convergence we have that

$$\left|\mu_t^i(\phi_j) - \mu_t(\phi_j)\right| \to 0, \quad \text{for a.e. } t \in [a,b].$$

Since $|\mu_t^i(\phi_j) - \mu_t(\phi_j)| \leq 2\Lambda$ it follows by the dominated convergence theorem that we can choose $I = I(\epsilon, J) > 0$ sufficiently large so that

$$\sum_{j=1}^{J} 2^{-j} \int_a^b \left|\mu_t^i(\phi_j) - \mu_t(\phi_j)\right| \, \mathrm{d}t < \epsilon. \tag{7.5}$$

In other words, $\mathcal{M}_i \to \mathcal{M}$ implies $d(\mathcal{M}_i, \mathcal{M}) \to 0$ by (7.4) and (7.5).

For the converse, suppose $d(\mathcal{M}_i, \mathcal{M}) \to 0$. Then for any fixed $j > 0$ we have that $2^j d(\mathcal{M}_i, \mathcal{M}) \to 0$ which implies

$$\int_a^b |\mu_t^i(\phi_j) - \mu_t(\phi_j)| \, \mathrm{d}t \to 0, \quad \text{for any } j > 0. \tag{7.6}$$

By Proposition 7.1.16, the integrand $f_i(t) = |\mu_t^i(\phi_j) - \mu_t(\phi_j)|$ is a continuous function of t for almost every $t \in [a,b]$. Further letting $Z_i \subset [a,b]$ denote the null-set of discontinuities of f_i, it follows that each f_i is continuous on $[a,b] \setminus Z$ where $Z = \cup_i Z_i$ is a null-set by countable sub-additivity. As such (7.6) implies $\mu_t^i(\phi_j) \to \mu_t(\phi_j)$ for almost every $t \in [a,b]$ and for each $j > 0$. Indeed if this were not the case we could find $\epsilon > 0$ and $t \in [a,b] \setminus Z$ such that $\liminf_{i\to\infty} f_i(t) \geq \epsilon > 0$. By the continuity of all f_i at t we can find $\delta > 0$ such that $\liminf_{i\to\infty} f_i(s) \geq \epsilon/2$ for each $s \in (t-\delta, t+\delta)$. This would contradict (7.6).

Now given any $\phi \in C_c^0(K)$ we can approximate $\hat{\phi} = \phi/|\phi|_\infty$ by a subsequence ϕ_{j_k}, and use that both $\mu_t(K), \mu_t^i(K)$ are bounded by Λ to show $\mu_t^i(\hat{\phi}) \to \mu_t(\hat{\phi})$ for a.e. $t \in [a,b]$. To see that this holds for almost every $t \in [a,b]$, note that the convergence holds precisely for $t \in [a,b] \setminus \cup_k Z_{j_k}$ where Z_{j_k} are the null sets of $t \in [a,b]$ where $\mu_t^i(\phi_{j_k})$ do not converge to $\mu_t(\phi_{j_k})$. By countable sub-additivity it follows that $\mathcal{L}^1(Z) = 0$.

□

Corollary 7.1.22. *Fix $a < b$ and an open subset $U \subset \mathbb{R}^{n+1}$. Let $K_i \subset\subset U$ satisfy $U = \cup_i K_i$, and $c_{K_i} \geq 1$. Consider the set of Brakke flows $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ satisfying $\mu_t(K_i) \leq c_{K_i}$ for each $t \in [a, b]$. The weak convergence of such Brakke flows is metrisable.*

Proof. For each $i > 0$ we can apply Proposition 7.1.21 to find a metric d_i for the flows restricted to K_i. Then we can define a metric by

$$d(\mathcal{M}, \mathcal{N}) = \sum_{i=1}^{\infty} 2^{-i} c_{K_i}^{-1} d_i(\mathcal{M}, \mathcal{N}).$$

□

Definition 7.1.23 (Brakke Convergence Metric)**.** Fix $a < b$ and an open subset $U \subset \mathbb{R}^{n+1}$. Let d denote the metric of Corollary 7.1.22.

Remark 7.1.24. Note that the specific metric is unimportant, any metrisation of Brakke flow convergence suffices for our purposes. Further note that for comparing self-shrinking flows it would suffice to define the metric for just the $t = -1$ slice of the flow. However as we will be studying flows that are only close to self-shrinkers, it is useful to have a metric for the convergence of a spacetime Brakke flow.

The following compactness theorem was proved by Ilmanen [Ilm94].

Theorem 7.1.25 (Compactness of Brakke Flows)**.** *Suppose $\mathcal{M}_i = \{\mu_t^i\}_{t\in[a,b]}$ is a sequence of integral Brakke flows on an open subset $U \subset \mathbb{R}^{n+1}$ and suppose we have the following uniform mass bound*

$$\sup_i \sup_{t\in[a,b]} \mu_t^i(K) \leq c_K < \infty \tag{7.7}$$

for any $K \subset\subset U$. Then we can find a subsequence so that $\mathcal{M}_i \to \mathcal{M}$ according to Definition 7.1.18.

Remark 7.1.26. In fact Ilmanen [Ilm94] also shows that for almost every t we can choose a further subsequence so that the varifolds V_t^i corresponding to μ_t^i also converge as varifolds to some V_t. However we will not make use of this fact.

7.2 Tangent Flows, Monotonicity and Singularities

The monotonicity formula for mean curvature flows is a key tool for studying tangent flows. In the smooth case the monotonicity formula is due to Huisken [Hui90], and it was extended to Brakke flows by Ilmanen [Ilm95] and White [Whi94]. An important assumption is that the initial Radon measure has uniformly bounded mass ratios. From this point on we will always assume the Brakke flows we work with satisfy this assumption.

Definition 7.2.1 (Uniform mass ratio bounds)**.** Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ denote an n-dimensional Brakke flow on $\Omega \subset \mathbb{R}^{n+1}$. We say $\mathcal{M}$ satisfies the uniform mass ratio bound if the following holds.

$$\sup\left\{\frac{\mu_t(B_r(x))}{\omega_n r^n} : x \in \Omega,\, t \in [a,b],\, 0 < r < \operatorname{dist}(x, \partial\Omega)\right\} < \infty. \tag{7.8}$$

Remark 7.2.2. It will follow from monotonicity that it suffices to have such a uniform mass ratio bound for just the initial time μ_a. Note that the assumption (7.7) necessary for compactness Theorem 7.1.25 follows from a uniform bound of (7.8) across a sequence of Brakke flows.

Compactness shows the existence of limit flows attained as subsequential limits of parabolic rescales of a Brakke flow. In particular tangent flows will be useful for studying the properties of singular points. We first describe parabolic rescales in spacetime.

Definition 7.2.3 (Spacetime)**.** Let $\mathbb{R}^{n+1,1}$ denote spacetime. Given $Y \in \mathbb{R}^{n+1,1}$ we write $Y = (y, s)$ where $y \in \mathbb{R}^{n+1}$ is a space variable and $s \in \mathbb{R}$ is a time variable.

Remark 7.2.4. Generally capital letters denoting points will denote spacetime points, for example $X = (x, t)$ and $Y = (y, s)$. There is no assumption that $t \geq 0$ or $s \geq 0$, however often there will be some initial and final time for the flow, in which case we may say $X \in \mathbb{R}^{n+1} \times [a, b]$.

The general idea of scaling in parabolic space is that a rescale by λ in space is related to a rescale in time by λ^2.

Definition 7.2.5 (Parabolic Rescale)**.** Given $Y = (y, s) \in \mathbb{R}^{n+1,1}$ and $s \in \mathbb{R}$ we define the parabolic rescale $D_{Y,\lambda} : \mathbb{R}^{n+1,1} \to \mathbb{R}^{n+1,1}$ by

$$D_{Y,\lambda}(x, t) = (\lambda(x - y), \lambda^2(t - s)).$$

Given a flow of surfaces $\mathcal{M} = \{M_t\}_{t\in[a,b]}$ we define

$$\mathcal{M}_{Y,\lambda} = \{D_{Y,\lambda}(M_t)\}_{t\in[a,b]}.$$

Given a Brakke flow $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ we define

$$\mathcal{M}_{Y,\lambda} = \{D_{Y,\lambda}\#(\mu_t)\}_{t\in[a,b]},$$

as the flow of push-forwards of the μ_t by $D_{Y,\lambda}$.

Remark 7.2.6. In other words we recentre on (y, s) then scale by λ in space, and λ^2 in time. The rescaled flows have a rescaled time parameter $\widetilde{t} = \lambda^2(t - s)$, such that $\widetilde{t} = 0$ corresponds to $t = s$.

If we consider $s > a$ and just times $a \leq t \leq s$ then the new time variable $\lambda^2(t - s)$ can take values between $\lambda^2(a - s)$ and 0. As $\lambda \to \infty$ this range extends to $(-\infty, 0]$.

It is important that the uniform mass ratio bounds of Definition 7.2.1 is preserved under these parabolic rescales and weak limits.

Proposition 7.2.7 (Preservation of Uniform Mass Ratio Bounds)**.** *Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ be an n-dimensional Brakke flow on $\Omega \subset \mathbb{R}^{n+1}$, and suppose for some $C > 0$ we have the following.*

$$\sup\left\{\frac{\mu_t(B_r(x))}{\omega_n r^n} : x \in \Omega,\, t \in [a, b],\, 0 < r < \operatorname{dist}(x, \partial\Omega)\right\} \leq C. \tag{7.9}$$

Then for any $Y = (y, s) \in \mathbb{R}^{n+1,1}$ and $\lambda > 0$ the parabolic rescale $D_{Y,\lambda}\mathcal{M}$ satisfies the same uniform mass ratio bound. Further for any sequence of $\mathcal{M}_i$ satisfying (7.9)*, if $\mathcal{M}_i$ converge as Brakke flows to $\mathcal{M}$, then $\mathcal{M}$ satisfies* (7.9) *also.*

Limit flows and tangent flows are then defined as follows.

Definition 7.2.8 (Limit Flows and Tangent Flows)**.** Let $\mathcal{M}$ denote a Brakke flow on $U \times [a, b]$ and suppose $X \in U \times [a, b]$. Then a limit flow at X is any flow $\mathcal{M}'$ achieved as the limit of $\mathcal{M}_{X_i,\lambda_i}$ for a sequence $X_i \in U \times [a, b]$ converging to X, and a sequence $\lambda_i > 0$ diverging to infinity $\lambda_i \to \infty$.

If the $X_i = X$ for all i we say $\mathcal{M}'$ is a tangent flow. Let $T_X\mathcal{M}$ denote the collection of all tangent flows to $\mathcal{M}$ at X.

Remark 7.2.9. In general it is unknown if $|T_X\mathcal{M}| = 1$, that is if tangent flows are unique.

The monotonicity formula is a key tool for studying the tangent flows of a Brakke flow. Originally the monotonicity formula was proved by Huisken [Hui90] for the mean curvature flow of smooth immersed surfaces. This was extended by Ilmanen [Ilm95] and White [Whi94] to more general Brakke flows. The proof in the smooth case can also be found in Ecker's book [Eck04].

The mass ratios of stationary varifolds satisfy a similar monotonicity formula. For Brakke flows we need to weight the mass by a Gaussian heat kernel $\Phi(x, r)$ which satisfies a useful partial differential equation, for example see the proof of the monotonicity formula in Ecker's book [Eck04].

Definition 7.2.10 (Gaussian Mass Ratio)**.** Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ be a Brakke flow of dimension n on $U \times [a, b]$. For any $x \in U$ and $r > 0$ define the Gaussian heat kernel as follows

$$\Phi_{x,r}(y) = (4\pi r^2)^{-\frac{n}{2}} \exp\left(-\frac{|y-x|^2}{4r^2}\right).$$

The Gaussian mass ratio at a point $X = (x, t) \in U \times [a, b]$ and scale $r > 0$ is defined as

$$\Theta_{\mathcal{M}}(X, r) = \int \Phi_{x,r} \mathrm{d}\mu_{t-r^2}.$$

Remark 7.2.11. The variable r controls both how far back in time we step from $X = (x, t)$, and how we weight the mass near X. For small r, the mass ratio $\Theta_{\mathcal{M}}(X, r)$ is mostly picking up mass from a small neighbourhood around x, at some short time before t.

The following generalisation of monotonicity to Brakke flows can be found in Ilmanen's paper [Ilm95]. In the smooth case where $M_t \subset \mathbb{R}^{n+1}$ is a mean curvature flow of surfaces the proof follows from the following identity satisfied by $\Psi(x, t) = \Phi_{0,t}(x)$, where div_{M_t} is the intrinsic divergence on M_t, Δ_{M_t} the intrinsic Laplacian, and $D^\perp$ is the gradient projected onto $T_x M_t^\perp$. Recall h is the mean curvature vector on M_t.

$$\frac{\partial \Psi}{\partial t} + \mathrm{div}_{M_t} D\Psi + \frac{|D^\perp \Psi|^2}{\Psi} = 0.$$

The result then follows by taking time derivatives of $\int_{M_t} \Psi \mathrm{d}\mathcal{H}^n$.

Theorem 7.2.12 (Monotonicity Theorem)**.** *Let $\mathcal{M} = \{\mu_t\}_{t\in[0,T)}$ be an n-dimensional Brakke flow on $U \times [0, T)$, and suppose the mass ratios are uniformly bounded as follows*

$$\sup_{x\in\mathbb{R}^{n+1}} \sup_{r>0} \frac{\mu_0(B_r(x))}{\omega_n r^n} \le C. \tag{7.10}$$

Then for any $X \in U \times [0, T)$ the Gaussian mass ratios satisfy the following for any $R \ge r > 0$.

$$\Theta_{\mathcal{M}}(X,R) - \Theta_{\mathcal{M}}(X,r) \geq \int_{t-R^2}^{t-r^2} \int \Phi_{X,\sqrt{t-s}}(y) \left| h(y,s) + \frac{(x-y)^\perp}{2(t-s)} \right| \mathrm{d}\mu_s(y)\,\mathrm{d}s$$

Here $(x-y)^\perp$ denotes the orthogonal projection onto the orthogonal complement of the approximate tangent plane $T_y\mu_s$.

In particular $\Theta_{\mathcal{M}}(X,r)$ is non-decreasing in r, and is constant if and only if the mean curvature vector satisfies

$$h(y,s) = -\frac{(y-x)^\perp}{2(t-s)}. \tag{7.11}$$

Remark 7.2.13. The equation (7.11) is known as the shrinker equation. A mean curvature flow is self-shrinking if and only if the mean curvature satisfies the shrinker equation. It is known by monotonicity that a flow is a self similar shrinker if and only if the Gaussian mass ratios are constant.

Since monotonicity is key to much of what follows, the initial uniform mass ratio bounds (7.10) will be assumed for any flow we work with. This bound is in effect saying that on any ball $B_r(x)$ the initial surface can only stack up so much in $B_r(x)$. For example for any $\delta > 0$ one can arrange that μ_0 is the mass measure of the planes $\mathbb{R}^n \times \{x_i\}$ with $|x_i| < \delta$ with $i = 1, \ldots, k$. In this case $\mu_0(B_1) \approx k$ for small δ. These initial mass ratio bounds can be extended across the whole flow by monotonicity.

Proposition 7.2.14 (Uniform Mass Ratio Bounds)**.** *Let $\mathcal{M}$ denote an n-dimensional Brakke flow on $B_1 \times [0,T)$ and suppose* (7.10) *holds. Then there is $\hat{C} = \hat{C}(n,T,\mathcal{M}) \geq C > 0$ such that*

$$\sup\left\{ \frac{\mu_t(B_r(x))}{\omega_n r^n} \,:\, (x,t) \in B_1 \times [0,T),\, 0 < r < \operatorname{dist}(x, \partial B_1) \right\} \leq \hat{C}. \tag{7.12}$$

Remark 7.2.15. Recall that (7.12) is preserved under parabolic rescales and limits of Brakke flows by Proposition 7.2.7.

Proof. This follows by using test functions to get a local version of monotonicity, for example as in Proposition 6.2 of Kasai-Tonegawa [KT14]. Then (7.12) follows from the initial assumption (7.10) and uniform bounds on $\mu_t(K)$ for $K \subset\subset B_1$. These uniform bounds on $\mu_t(K)$ follow from Proposition 7.1.15. □

By monotonicity we can define the Gaussian density, a mean curvature flow analogue of the density function for stationary varifolds, and energy density for stationary harmonic maps.

Definition 7.2.16 (Gaussian Density)**.** Given a Brakke flow $\mathcal{M}$ in $U \times [a, b]$ and $X \in U \times [a, b]$ the Gaussian density at X is defined by

$$\Theta_{\mathcal{M}}(X) = \lim_{r \to 0} \Theta_{\mathcal{M}}(X, r).$$

The following is a simple but useful relation between the Gaussian density and parabolic rescales and translations.

Proposition 7.2.17 (Gaussian Mass Ratios of Parabolic Rescales)**.** *Let $\mathcal{M}$ be a Brakke flow on $U \times [a, b]$ and $X \in U \times [a, b]$. Further let $Y \in \mathbb{R}^{n+1,1}$ and $\lambda > 0$. Then for any $r > 0$*

$$\Theta_{D_{Y,\lambda}\mathcal{M}}(D_{Y,\lambda}X, r) = \Theta_{\mathcal{M}}\left(X, \frac{r}{\lambda}\right).$$

Remark 7.2.18. In particular by letting $r \to 0$ we have that

$$\Theta_{D_{Y,\lambda}\mathcal{M}}(D_{Y,\lambda}X) = \Theta_{\mathcal{M}}(X).$$

Proof. By definition we have that

$$\Theta_{D_{Y,\lambda}\mathcal{M}}(D_{Y,\lambda}X, r) = \int \Phi_{\lambda(x-y),r}(z)\, \mathrm{d}\widetilde{\mu}_{\lambda^2(t-s)-r^2}(z).$$

Here $\widetilde{\mu}_t(A) = \mu_{t/\lambda^2+s}(\lambda A - y)$. The proof then follows by changing variables z to $\lambda(\widetilde{z} + y)$ and using the area formula. □

Brakke flows are continuous under spacetime translations, that is if $X_i \to X$ then $D_{X_i,1}\mathcal{M} \to D_{X,1}\mathcal{M}$. The continuity of spatial translation is simply due to the continuity of the test functions. The continuity of translations along the time axis follows as the mass measures μ_t are continuous in t for almost every time, Proposition 7.1.16, and because convergence of Brakke flows only requires convergence of the mass measures at almost every time.

Proposition 7.2.19 (Continuity of Spacetime translation and rescaling)**.** *Let $\mathcal{M}$ be a Brakke flow and suppose $X_i \in \mathbb{R}^{n+1,1}$ converge to X and $\lambda_i > 0$ converge to $\lambda > 0$. Then $D_{X_i,\lambda_i}\mathcal{M} \to D_{X,\lambda}\mathcal{M}$.*

Proof. We can split this into the cases $X_i = (x_i, 0)$ where $x_i \to 0$ with $\lambda_i = 1$, $X_i = (0, t_i)$ where $t_i \to 0$ and $\lambda_i = 1$, and $X_i = 0$ with $\lambda_i \to 1$ without loss of generality. In the first case we have that

$$D_{(x_i,0),1}\mu_t(\phi) = \int \phi(y - x_i)\, \mathrm{d}\mu_t = \mu_t(\phi_i).$$

Clearly $\phi_i(y) = \phi(y - x_i)$ converge to $\phi(y)$ pointwise, and for large i these are bounded by a function that is constant on some neighbourhood of $\operatorname{spt}(\phi)$ and 0 elsewhere, so by dominated convergence theorem $\mu_t(\phi_i) \to \mu_t(\phi)$.

In the second case $D_{(0,t_i),1}\mu_t = \mu_{t-t_i}$. Now by Proposition 7.1.16 we have that for almost every time t the measures converge $\mu_{t-t_i}(\phi) \to \mu_t(\phi)$.

In the final case we simply have that

$$D_{0,\lambda_i}\mu_t(\phi) = \int \phi(\lambda_i y)\, \mathrm{d}\mu_{\lambda_i^2 t} = \mu_{\lambda_i^2 t}(\phi_i), \quad \phi_i(y) = \phi(\lambda_i y).$$

This converges to $\mu_t(\phi)$ by the same arguments as above. □

Many of the properties of the density functions for stationary varifolds and stationary harmonic maps have analogues for the Gaussian density of mean curvature flows. For example we have the following upper-semicontinuity result. A proof can be found in Ecker's book [Eck04] and also White's paper [Whi97]. Note that this requires the uniform mass ratio bounds of Definition 7.2.1 or the initial bound (7.10), since the proof relies on monotonicity.

Lemma 7.2.20 (Upper-semicontinuity of Density)**.** *Let $\mathcal{M}_i$ be Brakke flows on $U \times [a,b]$ and $X_i \in U \times [a,b]$ such that $\mathcal{M}_i$ converge to a Brakke flow $\mathcal{M}$, and X_i to $X \in U \times [a,b]$. Further suppose $r_i \to 0$. Then*

$$\Theta_{\mathcal{M}}(X) \geq \limsup_{i\to\infty} \Theta_{\mathcal{M}_i}(X_i, r_i).$$

Remark 7.2.21. The result is also true when the right hand side is replaced by $\limsup_{i\to\infty} \Theta_{\mathcal{M}_i}(X_i)$, i.e. in the $r_i = 0$ case.

Recall that by Proposition 7.2.17 we have the following.

$$\Theta_{\mathcal{M}_{X,\lambda}}(0,r) = \Theta_{\mathcal{M}}\left(X, \frac{r}{\lambda}\right).$$

As such we have the following result about the Gaussian mass ratios of tangent flows.

Proposition 7.2.22 (Constancy of Gaussian Mass Ratios for Tangents)**.** *Suppose $\mathcal{N} \in T_X\mathcal{M}$ is a tangent flow to Brakke flow. Then $\mathcal{N}$ has constant Gaussian mass ratios at the origin satisfying*

$$\Theta_{\mathcal{N}}(0,r) = \Theta_{\mathcal{M}}(X), \quad r > 0.$$

By the shrinker equation (7.11), constancy of Gaussian mass ratios is equivalent to the flow being backward self-similar.

Definition 7.2.23 (Backward Self-Similar)**.** A Brakke flow $\mathcal{M}$ is said to be backward self similar if for any $t < 0$, and $\lambda > 0$ we have that $(D_{0,\lambda}\mathcal{M})\lfloor\{t < 0\} = \mathcal{M}\lfloor\{t < 0\}$.

Remark 7.2.24. In other words if surfaces $\Sigma_t \subset \mathbb{R}^{n+1}$ evolve by mean curvature flow, backwards self similarity means $\Sigma_{-1} = \lambda\Sigma_{-\lambda^2}$ for any $\lambda > 0$.

A flow that is backward self-similar is also called self-shrinking, and sometimes a shrinker, however this latter term is also frequently used to refer to just the time $t = -1$ slice of a backwards self-similar flow.

A flow being backwards self-similar is the analogous to a stationary harmonic map being conical, and a stationary varifold being a cone. In particular tangent flows are backward self-similar, and the Gaussian mass ratios are constant about the origin for backward self similar flows.

Lemma 7.2.25. *A Brakke flow $\mathcal{M}$ is backward self-similar if and only if the Gaussian mass ratios $\Theta_{\mathcal{M}}(0, r)$ are constant in $r > 0$.*

Corollary 7.2.26. *Any tangent flow $\mathcal{M}'$ to a Brakke flow $\mathcal{M}$ is backward self-similar.*

In some cases it is easy to compute the Gaussian density. By Lemma 7.2.25 it is clear that for a self shrinking Brakke flow we only need to compute $\Theta_{\mathcal{M}}(0, r)$ for some $r > 0$ to compute $\Theta_{\mathcal{M}}(0)$. The simplest example is when the flow is formed by some union of n-dimensional half planes, as the Gaussian kernel is normalised so that the integral over a density 1 plane is equal to 1. As such a union of k n-dimensional half planes in $\mathbb{R}^{n+1}$ meeting in equal angles along $\mathbb{R}^{n-1} \times \{0\}^2$ has Gaussian density $k/2$ at the origin.

Another simple computation is that the Gaussian density of a self shrinking circle $\mathcal{M}_t = S^1_{\sqrt{-2t}}$ is $\sqrt{2\pi/e} \in (3/2, 2)$. In particular the shrinking cylinder $\mathcal{M}_t = \mathbb{R}^{n-1} \times S^1_{\sqrt{-2t}}$ has the same density as the shrinking circle as the $\mathbb{R}^{n-1}$ directions simply integrate out.

Proposition 7.2.27 (Gaussian Densities)**.** *Let $H = \mathbb{R}^{n-1} \times [0, \infty) \times 0$ and V_k the varifold formed by k copies of H rotated by $\frac{2\pi}{k}$ about the axis $\mathbb{R}^{n-2} \times \{0\}^2$, and treated with density 1 along each rotated copy of H, except along the axis where the density is $k/2$. Then $\mathcal{M}_t = \|V_k\|$ for all $t \in \mathbb{R}$ is a well defined self-shrinking integral Brakke flow with Gaussian density at the origin $\Theta_{\mathcal{M}}(0) = k/2$.*

Let $\mathcal{N}_t = \mathbb{R}^{n-1} \times S^1_{\sqrt{-2t}}$ for $t < 0$ with density 1 along the circles. Then the Gaussian density at the origin of $\mathcal{N}$ is $\Theta_{\mathcal{N}}(0) = \sqrt{2\pi/e}$. Further if we let ψ_k denote the Gaussian density at the origin of the shrinking cylinder $\mathcal{N}_t = \mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ then

$$1 < \psi_0 < \psi_1 < \ldots < \psi_{n-1}.$$

Remark 7.2.28. The final statement was shown by Stone [Sto94]. Note in particular that the Gaussian density of the shrinking cylinder $\mathbb{R}^{n-1} \times S^1_{\sqrt{-2t}}$ is larger than the density of three n-dimensional half planes meeting along their boundary. This will be problematic later as we will want to rule out such stationary cones with density less that $\sqrt{2\pi/e}$. However note that this example of three half-planes is non-orientable, which will help us identify a class of flows where such stationary cones cannot arise as tangent flows.

The entropy of a surface was defined by Colding-Minicozzi [CM12] to study generic singularities, in particular showing that generic singularities are the only singularities for which the tangent flows are entropy-stable shrinkers. Shrinkers are critical points of a Gaussian weighted area functional by definition, but one needs to extend this functional to the entropy to get a useful definition of stability. We define the entropy in terms of these Gaussian area functionals $\mathcal{F}$.

Definition 7.2.29 ($\mathcal{F}$-functional and Entropy)**.** Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and define the n-dimensional $\mathcal{F}$-functional by

$$\mathcal{F}(\mu) = \frac{1}{(4\pi)^{n/2}} \int \exp\left(-\frac{|x|^2}{4}\right) \mathrm{d}\mu(x).$$

Given $\kappa > 0$ and $y \in \mathbb{R}^{n+1}$ let $\phi_{y,\kappa} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the spatial dilation and translation $\phi_{y,\kappa}(x) = y + \kappa x$. Then we define the entropy of μ by

$$\lambda(\mu) = \sup_{\kappa>0, y\in\mathbb{R}^{n+1}} \mathcal{F}((\phi_{y,\kappa})_{\#}\mu).$$

Remark 7.2.30. We can define λ for a surface Σ as simply the supremum of the Gaussian weighted masses of Σ over all dilations and translations of Σ. One can also interpret the translations and scalings in terms of re-centring and rescaling the Gaussian.

Note that if $\mathcal{M} = \{\mu_t\}_{t<0}$ defines a Brakke flow then

$$\mathcal{F}(\mu_{-1}) = \Theta_{\mathcal{M}}(0, 1).$$

If $\mathcal{M}$ is self-shrinking then $\Theta_{\mathcal{M}}(0,1) = \Theta_{\mathcal{M}}(0)$. In this case the $\mathcal{F}$-functional applied to the time $t=-1$ slice of a self-shrinking Brakke flow is precisely the Gaussian density at the origin of $\mathcal{M}$.

The $\mathcal{F}$-functionals and entropy satisfy bounds relative to the mass ratios of the Radon measure μ.

Proposition 7.2.31 (Entropy Bounds). *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ such that*

$$\sup_{y\in\mathbb{R}^{n+1},r>0} \frac{\mu(B_r(y))}{\omega_n r^n} < \infty. \tag{7.13}$$

Then we have the following $\mathcal{F}$-functional and entropy bounds.

$$\mathcal{F}(\mu) \leq \sup_{r>0} \frac{\mu(B_r)}{\omega_n r^n}, \quad \lambda(\mu) \leq \sup_{y\in\mathbb{R}^{n+1},r>0} \frac{\mu(B_r(y))}{\omega_n r^n}. \tag{7.14}$$

Remark 7.2.32. Recall that we assume any flow has a uniform mass ratio bound as in Proposition 7.2.14, in particular so that monotonicity applies. As such all flows we work with will have bounded entropy. Due to the preservation of uniform mass ratio bounds under weak limits this also implies that if $\mathcal{M}$ satisfies the mass ratio bounds with some constant C then so do all limit flows of $\mathcal{M}$.

Proof. The entropy bounds evidently follow from the $\mathcal{F}$-functional bounds. To bound $\mathcal{F}(\mu)$ note that by (7.13) we can apply integration by parts to get the following.

$$\mathcal{F}(\mu) = (4\pi)^{-\frac{n}{2}} \int \frac{r}{2} \exp\left(-\frac{r^2}{4}\right) \mu(B_r)\mathrm{d}r.$$

Writing $\frac{r}{2} = \frac{\omega_n r^{n+1}}{2}\frac{1}{\omega_n r^n}$ it follows that

$$\mathcal{F}(\mu) \leq \sup_{r>0}\left(\frac{\mu(B_r)}{\omega_n r^n}\right) \int_0^\infty \frac{\omega_n}{(4\pi)^{n/2}} \frac{r^{(n+1)/2}}{2} \exp(-r^2/4)\,\mathrm{d}r.$$

By using the change of variables $\sigma = r^2/4$ this gives the following bounds.

$$\mathcal{F}(\mu) \leq \sup_{r>0}\left(\frac{\mu(B_r)}{\omega_n r^n}\right) \frac{\omega_n}{\pi^{n/2}} \int_0^\infty \sigma^{n/2} e^{-\sigma}\,\mathrm{d}\sigma.$$

The integral $\int_0^\infty \sigma^{n/2}e^{-\sigma}\mathrm{d}\sigma$ is the Gamma function $\Gamma(\frac{n}{2}+1)$. Then the bound for $\mathcal{F}(\mu)$ in (7.14) follows from the formula

$$\omega_n = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}.$$

$\square$

The Huisken monotonicity Theorem 7.2.12 implies the entropy is monotone.

Proposition 7.2.33 (Monotonicity of Entropy)**.** *Let $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ denote an integral Brakke flow. Then $\lambda(\mu_t)$ is decreasing in t.*

We define the regular and singular sets of a flow as follows, following the definition in Kasai-Tonegawa [KT14], which updates Brakke's regularity [Bra78] by making use of Huisken's monotonicity [Hui90]. We will make use of the following Hölder norms.

Definition 7.2.34 (Hölder Norms)**.** Let $U \times (a,b) \subset \mathbb{R}^{n+1,1}$ be an open subset of spacetime, and suppose $f : U \to \mathbb{R}$. Given $X, Y \in U \times (a,b)$ define the parabolic norm

$$|X - Y|_{\mathcal{P}} = \max\left(|x-y|, |t-s|^{\frac{1}{2}}\right).$$

Define the parabolic Hölder norm as follows

$$|f|_{C^{0,\alpha}(U\times(a,b))} = \sup\left\{\frac{|f(X)-f(Y)|}{|X-Y|^{\alpha}_{\mathcal{P}}}, : X, Y \in U \times (a,b)\right\}.$$

Finally define the following parabolic time norm

$$|f|_{U\times(a,b),\alpha,\mathcal{P}} = \sup\left\{\frac{|f(y,t)-f(y,s)|}{|t-s|^{(1+\alpha)/2}} : y \in U,\ t,s \in (a,b)\right\}.$$

Remark 7.2.35. We can replace (a,b) by a closed interval $[a,b]$. Note that in the final parabolic time norm the space point y is fixed. This value is bounded above by $|f|_{C^{0,(1+\alpha)/2}(U\times(a,b))}$, but the regularity below only needs to bound the Hölder norm in the time variable, so it is useful to have this smaller value. If f is vector valued we replace $|f(X) - f(Y)|$ by the appropriate Euclidean norm.

Definition 7.2.36 (Regular and Singular Sets)**.** Given an n-dimensional Brakke flow $\mathcal{M} = \{\mu_t\}_{t\in[a,b]}$ on $U \times [a,b]$ and $\alpha > 0$ we say $X \in U \times [a,b]$ is a $C^{1,\alpha}$-regular point of $\mathcal{M}$ if the following holds.

A point $X = (x,t)$ with $x \in \operatorname{spt}(\mu_t)$ is said to be a $C^{1,\alpha}$-regular point if there is a neighbourhood V of X such that $V \cap \cup_{a\le t\le b}(\operatorname{spt}(\mu_t) \times \{t\})$ is the image of a map $f : B^n_R \times (t - R^2, t + R^2) \to V$ satisfying

$$|Df|_{C^{0,\alpha}(U\times[a,b])} < \infty,$$

$$|f|_{U\times[a,b],\alpha,\mathcal{P}} < \infty.$$

We denote by $\mathrm{Reg}(\mathcal{M})$ the collection of all such regular points. We denote by $\mathrm{Sing}(\mathcal{M})$ the collection of all points $X = (x,t)$ such that $x \in \mathrm{spt}(\mu_t) \setminus \mathrm{Reg}(\mathcal{M})$.

Remark 7.2.37. In other words, the regular points are points where the supports of μ_t are the image of a continuously differentiable spacetime function with parabolic Hölder continuous derivative, and an additional Hölder continuity on the time variable of the function.

The Brakke regularity theorem [Bra78] is an ϵ-regularity theorem for Brakke flows. Other such regularity theorems include Allard's regularity theorem [All72] for stationary varifolds, the Schoen-Uhlenbeck [SU82] regularity theorem for energy minimising maps and the Bethuel [Bet93] regularity theorems for stationary harmonic maps.

The statement of Brakke's regularity theorem given below is the partial regularity theorem as stated in Proposition 9.1 of Kasai-Tonegawa [KT14]. The theorem gives sufficient conditions for a point $X = (x,t)$ to be a regular point, in particular the flow is close to a unit density plane local to X in the sense that the L^2 height of the flow off this plane is very small.

One of the conditions of this theorem is that the Brakke flow has a local mass ratio less than that of a multiplicity two plane at some time just before t, and the flow hasn't disappeared entirely at some time just after t. These conditions prevent the flow from being close to a higher multiplicity plane or suddenly disappearing local to X. To quantify these conditions we need to fix a test function, as in Kasai-Tonegawa [KT14].

Let $\phi \in C^\infty([0,\infty))$ be a fixed test function such that $\phi(x) = 1$ for $x \in [0, (\frac{2}{3})^{\frac{1}{n}}]$, $\phi(x) = 0$ for $x \geq (\frac{5}{6})^{\frac{1}{n}}$ and $\phi(x) \in (0,1)$ otherwise. Further define $\phi_{x,R}(y) = \phi(\frac{|y-x|}{R})$ for $x, y \in \mathbb{R}^{n+1}$ and finally define

$$c_\phi = \int_{\mathbb{R}^n} \phi^2 \mathrm{d}\mathcal{H}^n.$$

With this test function fixed we can state the regularity theorem as follows.

Theorem 7.2.38 (Local Regularity Theorem)**.** *Suppose $\mathcal{M} = \{\mu_t\}_{t\in[0,T)}$ is an n-dimensional Brakke flow on $U \times [0,T)$ such that the varifolds V_t associated to each μ_t are unit density for almost every t. Further suppose we have the following uniform mass bound*

$$\frac{\mu_t(B_r(x))}{\omega_n r^n} \leq C, \quad \textit{for any } B_r(x) \subset U, t \in [0,T). \tag{7.15}$$

For every $\delta \in (0,1), \alpha \in (0,1)$ and $C > 0$ in (7.15) there is $L > 2$, $\Lambda > 3$ and $\epsilon \in (0,1)$ such that the following holds.

Let $X = (x,t) \in U \times (0,T)$ *and suppose for some* $R > 0$ *we have that*

$$B_{RL}(x) \times (t - R^2\Lambda, t + R^2\Lambda) \subset U \times (0,T).$$

Further suppose there is some plane $T \in G_n(n+1)$ *such that*

$$R^{-(k+2)} \int_{B_{RL}(x)} |T^{\perp}(y-x)|^2 \, \mathrm{d}\mu_{t-R^2\Lambda} < \epsilon^2. \tag{7.16}$$

Suppose there is $t_1 \in (t - R^2(\Lambda - \frac{5}{2}), s - R^2(\Lambda - 3))$ *and* $t_2 \in (s + R^2(\Lambda - 3), s + R^2(\Lambda - \frac{5}{2}))$ *such that*

$$\frac{\mu_{t_1}(\phi^2_{x,R})}{R^n} < (2-\delta)c_\phi, \quad \frac{\mu_{t_2}(\phi^2_{x,R})}{R^n} > \delta c_\phi. \tag{7.17}$$

Then $X = (x,t)$ *is a* $C^{1,\alpha}$*-regular point of* $\mathcal{M}$*.*

Remark 7.2.39. In fact Brakke [Bra78] and Kasai-Tonegawa [KT14] prove this in the arbitrary codimension case.

The assumption (7.16) implies that the flow is close to an n-dimensional plane at points close to x at some earlier time, in the sense that the L^2 height is very small.

The assumption (7.17) is saying that the flow cannot be close to a higher multiplicity plane local to x at some time just before t, and it cannot disappear suddenly at some time just after t. Due to monotonicity it suffices to have these properties at a single time before and after t.

The local regularity result gives the following result about the Hausdorff dimension and size of the singular set.

Corollary 7.2.40 (Qualitative Brakke regularity)**.** *Suppose* $\mathcal{M} = \{\mu_t\}_{t\in[0,T)}$ *is an* n*-dimensional Brakke flow on* $U \times [0,T)$ *such that the varifolds* V_t *associated to each* μ_t *are unit density for almost every* t*. Further suppose* $\mathcal{M}$ *satisfies* (7.15)*.*

Then for almost every $t \in (0,T)$ *there is a closed set* $S_t \subset \operatorname{spt}(\mu_t)$ *with* $\mathcal{H}^n(S_t) = 0$*, such that* $\operatorname{spt}(\mu_t) \setminus S_t \subset \operatorname{Reg}(\mathcal{M})$*.*

Note that unlike the regularity theorems for stationary varifolds and energy minimising maps, we do not have the result that $\Theta_{\mathcal{M}}(X) = 1$ implies X is a regular point. The issue is that a quasi-static plane is a well defined Brakke flow. This is defined by the surfaces $\mathcal{M}_t = \mathbb{R}^n \times \{0\}$ for all $t < 0$ and then $\mathcal{M}_t = \emptyset$ for all $t \geq 0$, i.e. the Radon measures associated to $\mathcal{M}_t$ for $t \geq 0$ are simply the null measures $\mu_t(A) = 0$ for any set $A \subset \mathbb{R}^{n+1}$. Due to the inequality in the definition of Brakke flow, this flow is still a Brakke flow, and $\Theta_{\mathcal{M}}(0) = 1$, since $\mathcal{M}_{-r^2}$ are unit density hyperplanes for all $r > 0$.

One way to deal with this issue is to work with some subclass of Brakke flows for which mass cannot disappear in this manner. A general such class are the unit regular Brakke flows, which is simply the class of Brakke flows for which unit density points are regular. This class can be shown to be closed under convergence of Brakke flows and is clearly closed under translation and parabolic rescales.

7.3 The singular set of a Brakke flow

The properties of a singularity $X \in \mathrm{Sing}(\mathcal{M})$ of a Brakke flow can be studied via the tangent flows $T_X\mathcal{M}$. We know that such tangent flows are self-similarly shrinking for negative times by Lemma 7.2.25. Any minimal cone can form a shrinking mean curvature flow by simply remaining stationary, for example as surfaces $\mathcal{M}_t = \mathbb{R}^k \times \{0\}^{n+1-k}$ for all $t \in \mathbb{R}$. Another minimal cone example is a union of 3 or more n-dimensional half planes meeting along an $(n-1)$-dimensional axis. In detail let

$$H = \mathbb{R}^{n-1} \times [0, \infty) \times \{0\},$$

and let $R_\theta(H)$ denote a rotation of H about $\mathbb{R}^{n-1} \times \{0\}^2$ by θ. Then for each $k \geq 3$ set

$$\Sigma_k = \bigcup_{i=1}^{k} R_{\frac{2\pi}{k}}(H).$$

This has a density function $\theta(x)$ which is 1 for all $x \in \Sigma_k \setminus \mathbb{R}^{n-1} \times \{0\}^2$ and equal to $k/2$ for $x \in \mathbb{R}^{n-1} \times \{0\}^2$. Then the varifold V_k formed by the set Σ_k and the density θ is stationary. As such $\mathcal{M}_t = V_k$ for all $t \in \mathbb{R}$ is a well defined self-shrinking Brakke flow, and it can easily be computed that $\Theta_{\mathcal{M}}(0) = k/2$. The case $k = 3$ is sometimes called a triple junction.

The standard example of non-stationary shrinkers are the cylinders $\mathcal{M}_t = \mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ for $k = 0, 1, \ldots, n$ and $t < 0$. There are also examples such as the Brakke spoon which is a non-compact non-embedded 1-dimensional shrinker, and a self-shrinking torus due to Angenent [Ang92]. The 1-dimensional compact self-shrinking curves were classified by Abresch-Langer [AL86], the shrinking circle being the only embedded example.

Similar to stationary harmonic maps and stationary varifolds the singular set of a Brakke flow can be stratified, though due to the parabolic nature of Brakke flow the statement and proof of such a result has some additional details. The stratification of the singular set of Brakke flows was proved by White [Whi97]. Its primary use for us will be to analyse the dimension of subsets of the singular set by studying the spine dimension of the tangent flows. Below we will outline the necessary details to state this result and make use of it.

Firstly we note that the Gaussian densities are maximal at the origin for self shrinking Brakke flows in the following sense.

Proposition 7.3.1 (Gaussian densities of shrinkers)**.** *Let $\mathcal{N}$ be a backwards self-similar Brakke flow, and $Y = (y, s) \in \mathbb{R}^{n+1,1}$ with $s \leq 0$. Then $\Theta_{\mathcal{N}}(Y) \leq \Theta_{\mathcal{N}}(0)$, with equality if and only if $\Theta_{\mathcal{N}}(Y, r)$ is independent of r.*

Remark 7.3.2. Note that $s \leq 0$ is necessary as we only know the $t < 0$ portion of $\mathcal{N}$ is invariant under parabolic dilations.

Proof. Since $\mathcal{N}$ is backwards self-similar we have that $D_{0,\lambda}\mathcal{N}$ equals $\mathcal{N}$ on the $t < 0$ parts. By Proposition 7.2.17 that for any $\lambda > 0$ and $r > 0$ the following holds.

$$\Theta_{\mathcal{N}}(Y, r) = \Theta_{D_{0,\lambda}\mathcal{N}}(Y, r) = \Theta_{\mathcal{N}}(D_{0,\lambda^{-1}}Y, r/\lambda).$$

Since $D_{0,\lambda^{-1}}(y, s) = (y/\lambda, s/\lambda^2)$ we have by upper semi-continuity Lemma 7.2.20 that for any sequence $\lambda_j \to \infty$ the following holds.

$$\Theta_{\mathcal{N}}(0) \geq \limsup_{j\to\infty} \Theta_{\mathcal{N}}(D_{0,\lambda^{-1}}Y, r/\lambda) = \Theta_{\mathcal{N}}(Y, r) \geq \Theta_{\mathcal{N}}(Y).$$

The final inequality follows from monotonicity. This proves the proposition since $\Theta_{\mathcal{N}}(0) \geq \Theta_{\mathcal{N}}(Y)$ with equality if and only if $\Theta_{\mathcal{N}}(Y, r)$ are constant. $\square$

The spine of a self-shrinking Brakke flow can be defined by the Gaussian density function in a manner analogous to stationary harmonic maps and stationary varifolds.

Definition 7.3.3 (Spine of a Shrinker)**.** Let $\mathcal{N}$ be a self shrinking Brakke flow. We define the spine of $\mathcal{N}$ as

$$\mathcal{S}(\mathcal{N}) = \{Y \in \mathbb{R}^{n+1,1} \,:\, \Theta_{\mathcal{N}}(Y) = \Theta_{\mathcal{N}}(0)\}.$$

The spine can be shown to be made up of a spatial spine and a temporal spine. The spatial spine corresponds to the spatial translations under which the $t < 0$ part of the shrinker is invariant. The temporal spine relates to translation invariance of the $t < 0$ part of the shrinker along the time axis. The temporal spine can either be just $\{0\}$, all times $\mathbb{R}$ or some subset of times $(-\infty, a]$ with $a \geq 0$. The idea is that a non-stationary shrinker is certainly not translation invariant along the time axis leading to the $\{0\}$ temporal spine, whereas a stationary cone is translation invariant along the time axis so long as it doesn't suddenly disappear at some time. This sudden disappearance of mass can only happen at a non-negative time as by definition of a self shrinker $\mathcal{M}_t = \frac{1}{\sqrt{-t}}\mathcal{M}_{-1}$ for all $t < 0$. This would mean we could translate the $t < 0$ part of the shrinker and it would be left invariant for any translation in time before the sudden mass disappearance, leading to the $(-\infty, a]$ for the temporal spine, and $\mathbb{R}$ case when there is no sudden mass disappearance.

Definition 7.3.4 (Spatial Spine)**.** Let $\mathcal{N}$ be a self shrinking Brakke flow. Then define the spatial spine as

$$\mathcal{S}_0(\mathcal{N}) = \{y \in \mathbb{R}^{n+1} : \Theta_{\mathcal{N}}((y,0)) = \Theta_{\mathcal{N}}(0)\}.$$

Remark 7.3.5. Clearly the spatial spine is just the time $t = 0$ slice of the full spine.

The $t < 0$ part of a self-shrinker is invariant under translations along the spatial spine as follows.

Proposition 7.3.6 (Translational invariance of shrinkers along spines)**.** *Let $\mathcal{N}$ be a self-shrinking Brakke flow . Then $\mathcal{S}_0(\mathcal{N}) \subset \mathbb{R}^{n+1}$ is a linear subspace. Further if $y \in \mathcal{S}_0(\mathcal{N})$ then $D_{(y,0),1}\mathcal{N} = \mathcal{N}$.*

Proof. Given $y \in \mathcal{S}_0(\mathcal{N})$ we have that $\lambda y \in \mathcal{S}_0(\mathcal{N})$ for any $\lambda > 0$ since

$$\Theta_{\mathcal{N}}((\lambda y, 0)) = \Theta_{D_{0,\lambda^{-1}}\mathcal{N}}((y,0)) = \Theta_{\mathcal{N}}((y,0)) = \Theta_{\mathcal{N}}(0).$$

We show $-y \in \mathcal{S}_0(\mathcal{N})$ by showing $\mathcal{N}$ is invariant under translation by y on the $t < 0$ part. To see this let $\lambda > 0$ be such that $\lambda^{-1} - \lambda = 1$. Let $\mathcal{N}_-$ denote the $t < 0$ part of the space time track. Then we have the following.

$$\mathcal{N}_- = D_{0,\lambda}\mathcal{N}_- = D_{(y,0),1}D_{(-y,0),1)}D_{0,\lambda}\mathcal{N}_-.$$

Now since $y \in \mathcal{S}_0(\mathcal{N})$ implies $\Theta_{\mathcal{N}}((y,0), r)$ is independent of r it follows that $\mathcal{N}$ is scale invariant about y, that is $D_{(y,0),\lambda^2}\mathcal{N}_- = D_{(y,0),1}\mathcal{N}_-$. Applying this then once more rescaling by λ^{-1} about the origin gives

$$\mathcal{N}_- = D_{(\lambda^{-1}-\lambda)y,0),1}\mathcal{N}_-.$$

Since $\lambda^{-1} - \lambda = 1$ by choice this implies $\mathcal{N}_-$ is invariant under translation by y. Similarly this argument can be used with $\lambda^{-1} - \lambda = -1$ to show $\mathcal{N}_-$ is translation invariant in direction $-y$, and so $-y \in \mathcal{S}_0(\mathcal{N})$ follows. Finally given $z \in \mathcal{S}_0(\mathcal{N})$ we clearly have that $\mathcal{N}_-$ is translation invariant in direction $y + z$, and so $\Theta_{\mathcal{N}}((y+z, 0)) = \Theta_{\mathcal{N}}(0)$. □

As described above, a non-stationary self-shrinker has no translation invariance along the time axis, and a stationary cone is translation invariant so long as mass doesn't suddenly disappear at some non-negative time. Since the spine can be thought of as directions of translation invariance, it follows that the spatial spine is a linear subspace of $\mathbb{R}^{n+1}$.

Proposition 7.3.7 (Translation invariance in time)**.** *Given a self shrinking Brakke flow $\mathcal{N}$ for $T \geq 0$ there is a set $A \subset \mathbb{R}$ such that $\mathcal{S}(\mathcal{N}) = \mathcal{S}_0(\mathcal{N}) \times A$ for three cases $A = \{0\}$ when $\mathcal{N}$ is non-stationary, $A = \mathbb{R}$ when $\mathcal{N}$ is stationary and mass doesn't disappear, or $A = (-\infty, a]$ for some $a \geq 0$ when $\mathcal{N}$ is quasi-static, that is a stationary cone that suddenly disappears at time a.*

Remark 7.3.8. White [Whi97] gives an example of a 1-dimensional Brakke flow formed of two circles with different radii connected by a line segment. The tangents to singularities of this flow can be easily visualised and an example of each of the different situations for the spine described above can be found at points on this flow.

We define the following parabolic spine dimension. Full translation invariance along the time axis counts for 2 dimensions. Quasi-static flows do not get the additional two dimensions from partial translation invariance in time. An easy way to distinguish this is to define the parabolic spine dimension dependent on the set A described in Proposition 7.3.7.

Definition 7.3.9 (Parabolic Spine Dimension)**.** Let $\mathcal{N}$ denote a backwards self-similar Brakke flow with a spine $\mathcal{S}(\mathcal{N}) = \mathcal{S}_0(\mathcal{N}) \times A$, we define $d = \dim(\mathcal{S}_0(\mathcal{N}))$ and $D(\mathcal{N}) = d$ if $A \neq \mathbb{R}$ and $D(\mathcal{N}) = d + 2$ if $A = \mathbb{R}$.

Using this we define parabolic strata of the singular set.

Definition 7.3.10 (Singular Strata)**.** Let $\mathcal{M}$ be an integral Brakke flow on $U \times [a, b]$. We define for $k = 0, 1, \ldots, n + 2$ the k-th singular strata as follows.

$$\mathrm{Sing}_k(\mathcal{M}) = \{Y \in \mathrm{Sing}(\mathcal{M}) \,:\, D(\mathcal{N}) \leq k \text{ for all } \mathcal{N} \in T_Y\mathcal{M}\}.$$

The stratification result of White [Whi97] is as follows.

Theorem 7.3.11 (Stratification of Singular Set)**.** *Let $\mathcal{M}$ be an integral Brakke flow. Then the parabolic Hausdorff dimension of* $\mathrm{Sing}_k(\mathcal{M})$ *is bounded by k.*

$$\dim_{para}(\mathrm{Sing}_k(\mathcal{M})) \leq k.$$

7.4 Cylindrical Singularities

A singularity $X \in \mathrm{Sing}(\mathcal{M})$ is said to be cylindrical if some tangent flow at X is a shrinking cylinder. Certain results are known about cylindrical singularities that are unknown for a general singularity.

Definition 7.4.1 (Cylindrical Singularity)**.** Let $\mathcal{M}$ denote a Brakke flow and $X \in \mathrm{Sing}(\mathcal{M})$. We say X is a cylindrical singularity if there is a shrinking cylinder as a tangent flow to X, that is some rotation of the flow $\mathcal{N}_t = \mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ is in the tangent space to X, $\mathcal{N} \in T_X\mathcal{M}$.

Remark 7.4.2. Below we will see that by [CM15] cylindrical tangents are unique, and so an equivalent definition is that X is a cylindrical singularity if all tangents are the same shrinking cylinder.

Note that we will always be referring to multiplicity one shrinking cylinders. One could define a self shrinking Brakke flow by a multiplicity 2 or larger shrinking cylinder, but we will not refer to this as a shrinking cylinder, as many of the results below would not apply in this case.

The cylindrical singularities are of particular interest for a number of reasons. A mean convex surface is a surface with non-negative mean curvature. This property is preserved by mean curvature flow. In White's papers [Whi03] and [Whi15] it was shown that the only singularities that an initially mean-convex surface develops under mean curvature flow are cylindrical singularities.

One property is that the class of shrinking cylinders is compact, due to the compactness of $G_k(n)$. This is a simple result, however it is useful to state to draw analogies to the method for stationary harmonic maps.

Proposition 7.4.3 (Compactness of Cylindrical Class)**.** *Let $\mathcal{N}_i$ denote a sequence of self-shrinking cylinders. Then there is a subsequence such that $\mathcal{N}_i$ converge weakly to a self-shrinking cylinder.*

Proof. First choose a subsequence on which all the $\mathbb{R}^k$ factors are the same dimension. Then choose a subsequence so that the rotations of these factors are converging to some fixed rotation. Without loss of generality suppose this fixed rotation is simply the identity. Then it is easy to show by using test functions that $\mathcal{N}_i$ are converging weakly to $\mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ for $t < 0$. □

It is clear by computation or by observation of the translation invariance that the spine of a shrinking cylinder is the axis of rotational symmetry of the cylinder. As such the singular set of a shrinking cylinder flow is equal to the spine of the shrinking cylinder. In the stationary harmonic map case we needed a maximal spine dimension for such a result.

Proposition 7.4.4 (Spine of a Cylinder). *Let $\mathcal{N} = \{\nu_t\}_{t<0}$ denote the shrinking cylinder where ν_t are supported on $\mathbb{R}^{n-k} \times S^k_{\sqrt{-2(n-k)t}}$. Then the spine of $\mathcal{N}$ is given by*

$$\mathcal{S}(\mathcal{N}) = \left(\mathbb{R}^{n-k} \times \{0\}^{k+1}\right) \times \{0\}.$$

Remark 7.4.5. Note that the spine here is the spacetime subspace defined by setting the last $(k+1)$-spatial coordinates and the time coordinate to zero, that is $x_{n-k+1}, \ldots, x_{n+1} = 0$ and $t = 0$.

It was shown by Colding-Ilmanen-Minicozzi [CIM15] that all tangent flows at a cylindrical singularity are shrinking cylinders with the same spine dimension, though possibly with different axes. Part of this proof requires a rigidity theorem that roughly says a shrinker that is weakly close to a shrinking cylinder is also a shrinking cylinder. A rough outline of the method of proof is to show that if a shrinker is near a cylinder on some scale, then it is also near a cylinder on some larger scale, but the estimate will become slightly weaker. Then through an iterative step, on a large enough scale one can re-improve the estimate.

Later it was shown by Colding-Minicozzi [CM15] that the spine of the shrinking cylinder tangent is also unique. The method here makes use of an infinite dimensional Lojasiewicz inequality, and in particular the method has to deal with the fact that shrinking cylinders are non-compact, and that only part of the original flow is a graph over this tangent. Simon [Sim83a] originally made use of a Lojasiewicz inequality to study asymptotics for evolution equations, in particular proving a uniqueness result for tangent maps to energy minimising maps at isolated singularities with a smooth tangent, and an analytic target.

First we state the rigidity result of Colding-Ilmanen-Minicozzi [CIM15].

Theorem 7.4.6 (Rigidity Theorem for Shrinking Cylinders). *Given $n \geq 1$, $\lambda_0, C > 0$ there is $R = R(n, \lambda_0, C) > 0$ such that the following holds. If $\Sigma \subset \mathbb{R}^{n+1}$ is an n-dimensional shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$, and Σ is smooth in B_R with mean curvature $H_\Sigma \geq 0$ and second fundamental form $|A_\Sigma| \leq C$ on $B_R \cap \Sigma$, then Σ is a rotation of $\mathbb{R}^{n-k} \times S^k$ for some $k \leq n$.*

Remark 7.4.7. Note that by Proposition 7.2.31 it suffices that the area ratios of Σ are uniformly bounded by λ_0 for the required entropy bound. Recall that such we are assuming any flow we work with has uniformly bounded mass ratios as in Proposition 7.2.14.

This can be extended to self-shrinking Brakke flows by Allard regularity [All72]. This result is Corollary 2.12 of Colding-Ilmanen-Minicozzi [CIM15]. Recall from Definition 7.1.23 that d is some metrisation of the weak convergence of Brakke flows.

Corollary 7.4.8 (Rigidity under weak convergence)**.** *Given $n, k, \lambda_0 > 0$ there is $\delta = \delta(n, k, \lambda_0) > 0$ such that the following holds. Suppose $\mathcal{M} = \{\mu_t\}_{t\in\mathbb{R}}$ is a self-shrinking integral Brakke flow such that $\lambda(\mu_{-1}) \leq \lambda_0$. Further suppose $\mathcal{N}$ is a shrinking cylinder with spine dimension $n-k$, and $d(\mathcal{M}, \mathcal{N}) < \delta$. Then $\mathcal{M}$ is a small rotation of $\mathcal{N}$, in particular $\mathcal{M}$ is also a shrinking cylinder with spine dimension $n-k$.*

The result of Colding-Minicozzi [CM15] builds on this to show that at a cylindrical singularity, any other tangent is not just a shrinking cylinder of the same spine dimension, but also with the same axis. In other words, tangents at cylindrical singularities are unique.

Theorem 7.4.9 (Uniqueness of Cylindrical Tangents)**.** *Let $\mathcal{M}$ denote an integral Brakke flow and suppose $X \in \operatorname{Sing}(\mathcal{M})$ is a cylindrical singularity with spine dimension k. Then any other tangent flow at X is the same shrinking cylinder.*

Chapter 8

The Structure Theorem for Mean Curvature Flows

8.1 Overview

In this chapter we present the key steps to show that for a singularity $X \in \mathrm{Sing}(\mathcal{M})$ with a shrinking cylinder tangent $\mathcal{N} \in T_x\mathcal{M}$, the collection of singularities $Y \in \mathrm{Sing}(\mathcal{M})$ with $\Theta_{\mathcal{M}}(Y) \geq \Theta_{\mathcal{M}}(X)$ also have shrinking cylinders as tangents local to X. To show this we approximate the rescales $D_{Y,\lambda}\mathcal{M}$ by shrinking cylinder flows for all Y sufficiently close to X with $\Theta_{\mathcal{M}}(Y) \geq \Theta_{\mathcal{M}}(X)$ and $\lambda > 0$ sufficiently large.

In section 8.2 we show that for any singularity $X \in \mathrm{Sing}(\mathcal{M})$ we can find backwards self-similar flows that well approximate $D_{Y,\lambda}\mathcal{M}$ for Y sufficiently close to X and $\lambda > 0$ sufficiently large. We will call these pseudo-tangent flows.

The aim is now to show that if $X \in \mathrm{Sing}(\mathcal{M})$ has a shrinking cylinder as a tangent, then by the rigidity theorem of Colding-Ilmanen-Minicozzi [CIM15] the pseudo-tangent flows are also shrinking cylinders. To use this result we compare a pseudo-tangent flow at Y with scale $\lambda > 0$ to the rescale $D_{Y,\lambda}\mathcal{M}$, and the shrinking cylinder tangent at X to some rescale $D_{X,R}\mathcal{M}$ for sufficiently large $R > 0$. Both of these comparisons are good by construction, and so we only need to show the two rescales $D_{X,R}\mathcal{M}$ and $D_{Y,\lambda}\mathcal{M}$ are close for Y sufficiently close to X, and λ sufficiently close to R. This is a continuity result for the parabolic rescaling of Brakke flows with respect to the convergence of Brakke flows. Such a result is proved in section 8.3.

In section 8.4 we use an iterative argument to find a uniform radius around a cylindrical singularity $X \in \mathrm{Sing}(\mathcal{M})$ at which we can show pseudo-tangents at $Y \in \mathrm{Sing}(\mathcal{M})$ with $\Theta_{\mathcal{M}}(Y) \geq \Theta_{\mathcal{M}}(X)$ are also shrinking cylinders for all sufficiently large scales. As such this implies there is a shrinking cylinder as a tangent to such points

Y. The method of Colding-Minicozzi [CM16] can then be applied to give a structure result for this set of singularities.

Finally in section 8.5 we show that for the class of Brakke flows arising from elliptic regularisation, we can rule out particular tangents at points $Y \in \mathrm{Sing}(\mathcal{M})$ close to a cylindrical singularity $X \in \mathrm{Sing}(\mathcal{M})$ with $(n-1)$-dimensional spine, and such that $\Theta_{\mathcal{M}}(Y) < \Theta_{\mathcal{M}}(X)$. The main idea here is to rule out triple junction type behaviour that can have lower Gaussian density than X, but appear in higher dimensional singular strata.

8.2 Pseudo-tangent flows

We wish to define self-shrinking pseudo-tangent flows on a subset of the singular set. The pseudo-tangents will be good approximations of the parabolic rescales of the flow, quantified using the metric of Definition 7.1.23 associated to convergence of Brakke flows. However we can only show these pseudo-tangents exist and are self-shrinking on a subset of the singular set.

Given a Brakke flow $\mathcal{M}$ and a singularity $X \in \mathrm{Sing}(\mathcal{M})$, we split $\mathrm{Sing}(\mathcal{M})$ up into points with density less than $\Theta_{\mathcal{M}}(X)$ and those with density at least $\Theta_{\mathcal{M}}(X)$. The latter set we call $S^+(X)$.

Throughout this section we fix a Brakke flow $\mathcal{M}$ which exists for times $t \in [0, T)$. Further we fix a singularity $X = (x, t) \in \mathrm{Sing}(\mathcal{M})$ with $t > 0$. Any dependencies of constants on these will be made clear as they arise.

Definition 8.2.1 ($S^+(X)$)**.** Let $\mathcal{M}$ be a Brakke flow and suppose $X \in \mathrm{Sing}(\mathcal{M})$. We define

$$S^+(X) = \{Y \in \mathrm{Sing}(\mathcal{M}) \,:\, \Theta_{\mathcal{M}}(Y) \geq \Theta_{\mathcal{M}}(X)\}.$$

Note that by upper semicontinuity of the Gaussian density that locally $S^+(X)$ consists of singularities with density just greater than $\Theta_{\mathcal{M}}(X)$. As such if $Y_i \in S^+(X)$ and $Y_i \to X$, then $\Theta_{\mathcal{M}}(Y_i) \searrow \Theta_{\mathcal{M}}(X)$. Further we have that $S^+(X)$ is a closed set.

Proposition 8.2.2 ($S^+(X)$ is closed)**.** *Let $\mathcal{M}$ be a Brakke flow and $X \in \mathrm{Sing}(\mathcal{M})$. Then $S^+(X)$ is a closed set.*

Proof. This follows immediately by upper-semicontinuity, if $X_i \in S^+(X)$ converge to Y then

$$\Theta_{\mathcal{M}}(Y) \geq \limsup_{i\to\infty} \Theta_{\mathcal{M}}(X_i) \geq \Theta_{\mathcal{M}}(X).$$

□

We will use the following convention for a parabolic ball.

Definition 8.2.3 (Parabolic Ball)**.** For any $r > 0$ we define the parabolic ball $P_r(X) = B_r(x) \times (t - r^2, t + r^2)$ where $X = (x, t)$.

Remark 8.2.4. In other words $P_r(X)$ is the ball of radius r centred on X according to the parabolic norm $|(x,t)| = \max(|x|, \sqrt{|t|})$. Note that $P_r(X)$ contains points (y, s) with time $s > t$. In some cases this convention is not used, and the parabolic balls only go backwards in time from the base point.

We first prove that limit flows along $S^+(X)$ are self-shrinking flows, with Gaussian density at the origin equal to $\Theta_{\mathcal{M}}(X)$. The key part of the proof is that points in $S^+(X)$ have a useful density lower bound.

Lemma 8.2.5 (Limit flows on $S^+(X)$)**.** *Let $\mathcal{M}$ be an integral Brakke flow, $X \in \mathrm{Sing}(\mathcal{M})$ and suppose $Y_i \in S^+(X)$ converge to X. For any sequence $\lambda_i \to \infty$ we can find a subsequence of $\mathcal{M}_i = D_{Y_i,\lambda_i}\mathcal{M}$ that converges to a self-shrinker $\mathcal{M}'$. Further $\Theta_{\mathcal{M}'}(0) = \Theta_{\mathcal{M}}(X)$.*

Proof. Recall we are assuming $\mathcal{M}$ has some uniform mass ratio bound as in Definition 7.2.1. Since $\mathcal{M}_i = D_{Y_i,\lambda_i}\mathcal{M}$ is a sequence of rescales there is some mass ratio bound that is uniform across the $\mathcal{M}_i$. As such compactness Theorem 7.1.25 applies. We can find a subsequence so that $\mathcal{M}_i$ converge to some $\mathcal{M}'$. By upper semicontinuity, Lemma 7.2.20 and by definition of $S^+(X)$ we have that

$$\Theta_{\mathcal{M}'}(0) \geq \limsup_{i\to\infty} \Theta_{\mathcal{M}_i}(0) = \limsup_{i\to\infty} \Theta_{\mathcal{M}}(Y_i) \geq \Theta_{\mathcal{M}}(X).$$

For almost every $r > 0$, and again using upper semicontinuity we have that

$$\Theta_{\mathcal{M}'}(0, r) = \lim_{i\to\infty} \Theta_{\mathcal{M}_i}(0, r) \leq \limsup_{i\to\infty} \Theta_{\mathcal{M}}\left(Y_i, \frac{r}{\lambda_i}\right) \leq \Theta_{\mathcal{M}}(X).$$

Now by monotonicity Theorem 7.2.12, $\Theta_{\mathcal{M}'}(0, r)$ is constant for all r, and so $\mathcal{M}'$ is backward self similar by Lemma 7.2.25. $\square$

Using the metric d of Definition 7.1.23 we can state a quantitative version of this result. In the following we assume the metric d is defined to compare Brakke flows that exist for times on some superset of $[-1, 0)$. This will suffice for our purposes later as we ultimately wish to compare two self-shrinking flows.

Corollary 8.2.6 (Existence of Pseudo-Tangents)**.** *Suppose $\mathcal{M}$ is an integral Brakke flow for $t \in [0, T)$, and $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$. Then for any $\epsilon > 0$ there is $r = r(\mathcal{M}, X_0, \epsilon, n) \in (0, 1)$ such that for any $Y \in S^+(X) \cap P_r(X_0)$ and $\lambda > \frac{1}{r}$ there exists a self-shrinking flow $\mathcal{N} = \{\nu_t\}_{t\in(-\infty,0]}$ such that $\Theta_{\mathcal{N}}(0) = \Theta_{\mathcal{M}}(X_0)$ and*

$$d(\mathcal{N}, D_{Y,\lambda}\mathcal{M}) < \epsilon. \tag{8.1}$$

Remark 8.2.7. Note that $D_{Y,\lambda}\mathcal{M}$ exists for times in $[-1, 0)$ for sufficiently large λ dependent only on $t_0 > 0$. This dependence is covered by the fact r depends on X_0. It is worth noting this is why we need $t_0 > 0$, so that the rescales are eventually defined for times in $[-1, 0)$. This wouldn't be possible if $X_0 = (x, 0)$ unless we extended the flow $\mathcal{M}$ backwards in time.

Proof. This follows from Lemma 8.2.5. Indeed if the corollary were false, we would have a sequence Y_i and λ_i exactly as in Lemma 8.2.5, but for which $d(\mathcal{N}, D_{Y_i,\lambda_i}\mathcal{M})$ is at least ϵ for all self-shrinking flows $\mathcal{N}$ with $\Theta_{\mathcal{N}}(0) = \Theta_{\mathcal{M}}(X)$. However this contradicts Lemma 8.2.5 which concludes that a subsequence of $D_{Y_i,\lambda_i}\mathcal{M}$ must converge to such a shrinker. □

8.3 Continuity of Rescaling

In this section we will show that translation along $S^+(X)$ and parabolic rescaling around points in $S^+(X)$ is a continuous transformation with respect to the metric of weak convergence of Brakke flows. This is analogous to the continuity results for stationary harmonic maps. This result allows us to compare pseudo-tangent flows to each other by comparing the parabolic rescales of the underlying flow $\mathcal{M}$.

This is an extension of Proposition 7.2.19. We now want to say the distance $d(D_{Y,\lambda}\mathcal{M}, D_{Z,\kappa}\mathcal{M})$ is small for arbitrarily large $\lambda, \kappa > 0$, under suitable conditions. The proof of the following result is by contradiction, making use of Lemma 8.2.5.

Lemma 8.3.1. *Let $\mathcal{M}$ be an integral Brakke flow for $t \in [0, T)$ and $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$. For any $\epsilon > 0$ there is $\delta = \delta(\mathcal{M}, X_0, \epsilon, n) > 0$ such that the following holds. Suppose $\lambda, \kappa > \delta^{-1}$, $\frac{\lambda}{2} \leq \kappa \leq 2\lambda$ and $Y, Z \in S^+(X_0) \cap P_{\lambda^{-1}}(X_0)$. Then*

$$d(D_{Y,\lambda}\mathcal{M}, D_{Z,\kappa}\mathcal{M}) < \epsilon.$$

Remark 8.3.2. Again note that the metric d is only comparing the flows for times $t \in [-1, 0)$. The fact that $t < 0$ is important in the proof so that $d(\mathcal{M}, \mathcal{N}) = 0$ if

$\mathcal{M}$ and $\mathcal{N}$ are equal on their $t < 0$ parts. In particular a quasi-static flow of some stationary cone is equal in the metric to a static flow of the same cone, as they are equal on their $t < 0$ parts. In this sense the metric is only a metric on the parts of the flows with times $t \in [-1, 0)$.

Proof. Suppose this were not the case and a consider sequence $Y_i, Z_i \in S^+(X_0) \cap P_{\lambda_i^{-1}}(X_0)$ and $\lambda_i, \kappa_i > i$ such that $\lambda_i/\kappa_i \in [\frac{1}{2}, 2]$, but

$$d(D_{Y_i,\lambda_i}\mathcal{M}, D_{Z_i,\kappa_i}\mathcal{M}) \geq \epsilon.$$

By Lemma 8.2.5 we can take a subsequence such that $\lambda_i/\kappa_i \to \tau \in [\frac{1}{2}, 2]$, and $D_{Y_i,\lambda_i}\mathcal{M}, D_{Z_i,\kappa_i}\mathcal{M}$ converge to self-shrinkers $\mathcal{N}_1, \mathcal{N}_2$ respectively. Further we have that

$$d(\mathcal{N}_1, \mathcal{N}_2) \geq \epsilon.$$

Let $Y_i = (y_i, s_i)$ and $Z_i = (z_i, u_i)$ and note that since $Y_i, Z_i \in P_{\lambda_i^{-1}}(X_0)$ we have that $\lambda_i(y_i - z_i)$ and $\lambda_i^2(s_i - u_i)$ are both bounded sequences. Then we can choose a subsequence so that $W_i = (\lambda_i(y_i - z_i), \lambda_i^2(s_i - u_i))$ converge to some $W = (w, s)$. Since $\lambda_i/\kappa_i \to \tau > 0$ we can apply Proposition 7.2.19 to show the following

$$D_{Y_i,\lambda_i}\mathcal{M} = D_{W_i,\lambda_i/\kappa_i}(D_{Z_i,\kappa_i}\mathcal{M}) \to D_{W,\tau}(\mathcal{N}_2).$$

As such since $D_{Y_i,\lambda_i}\mathcal{M} \to \mathcal{N}_1$ we have that $\mathcal{N}_1 = D_{W,\tau}\mathcal{N}_2$. We can use the Proposition 7.2.17 and the fact that $Y_i \in S^+(X_0)$ to give the following.

$$\Theta_{\mathcal{M}}(X_0) \leq \Theta_{\mathcal{M}}(Y_i) = \Theta_{D_{Y_i,\lambda_i}\mathcal{M}}(0) = \Theta_{D_{Z_i,\kappa_i}\mathcal{M}}(W_i).$$

As such by upper semicontinuity we have that

$$\Theta_{\mathcal{N}_2}(W) \geq \Theta_{\mathcal{M}}(X_0) = \Theta_{\mathcal{N}_2}(0).$$

This implies that $W \in \mathcal{S}(\mathcal{N}_2)$. As such $\mathcal{N}_2 \lfloor \{t < 0\}$ is invariant under parabolic rescales centred on W. This implies that $\mathcal{N}_2 \lfloor \{t < 0\}$ is equal to $\mathcal{N}_1 \lfloor \{t < 0\}$, contrary to $d(\mathcal{N}_1, \mathcal{N}_2) \geq \epsilon > 0$.

$\square$

Note that in the final step we require that if $\mathcal{N}_1 \lfloor \{t < 0\} = \mathcal{N}_2 \lfloor \{t < 0\}$ then $d(\mathcal{N}_1, \mathcal{N}_2) = 0$. This wouldn't be true if the metric also compared positive times, for example $\mathcal{N}_1$ could be a static plane and $\mathcal{N}_2$ a quasi-static plane that disappears at time $t = 0$. Then clearly any metric that compared $\mathcal{N}_1$ and $\mathcal{N}_2$ at times $t > 0$ would not be 0. In fact we can avoid this situation under the later assumption that there is a shrinking cylinder tangent at X_0. In this case we can take $Z_i = X_0$ in the proof above, then by the uniqueness of cylindrical tangents we could show $\mathcal{N}_2$ is a shrinking cylinder, so $W \in \mathcal{S}(\mathcal{N}_2)$ would be $W = (w, 0)$ with w on the axis of the cylinder.

Using this result with the rigidity result Theorem 7.4.6 and its Corollary 7.4.8, we can show that the pseudo-tangents are shrinking cylinders at points in $S^+(X_0)$ sufficiently close to a cylindrical singularity X_0. This will be true for the pseudo-tangents at all sufficiently large scales λ, and as such we can then use the rigidity again to show the tangent flows in $S^+(X_0)$ are cylinders for points sufficiently close to X_0. From there one can apply the method of Colding-Minicozzi [CM16] to $S^+(X_0)$ local to X_0 to acquire a structure result for $S^+(X_0)$.

There is one issue to do with the scales. Note that $Y, Z \in P_{\lambda^{-1}}(X_0)$ in Lemma 8.3.1, and λ is the scale factor of the parabolic rescale about Y. We want to remove this codependency and find a scale dependent only on $\mathcal{M}$ and the cylindrical singularity X_0. This can be achieved by means of an iteration argument in the next section.

8.4 Rigidity of $S^+(X_0)$ at cylindrical singularities

In this section we suppose $\mathcal{M}$ is an integral Brakke flow for $t \in [0, T)$, and $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$ has a cylindrical tangent $\mathcal{C} \in T_{X_0}\mathcal{M}$. As such by Theorem 7.4.9 this is the unique tangent at X_0. Further by the rigidity result Theorem 7.4.6 and its Corollary 7.4.8, any shrinker that is sufficiently weakly close to this tangent will also be a shrinking cylinder with the same spine dimension.

In the following theorem we aim to find a parabolic radius δ around X_0 such that the pseudo-tangent flows $\mathcal{N}$ of Corollary 8.2.6 at points $Y \in S^+(X_0) \cap P_\delta(X_0)$ are shrinking cylinders at sufficiently large scales $\lambda > \delta^{-1}$. Let $0 \leq k \leq n$ denote the spine dimension of $\mathcal{C} \in T_{X_0}\mathcal{M}$. The rigidity result Corollary 7.4.8 gives some $\epsilon_* = \epsilon_*(\lambda_0, n, k) > 0$ such that if $\mathcal{N}$ is a shrinker and $\mathcal{C}$ a shrinking cylinder, and $d(\mathcal{N}, \mathcal{C}) < \epsilon_*$ then $\mathcal{N}$ is also a shrinking cylinder. In fact we only need that the time -1 slices are close. The λ_0 here is a uniform entropy bound for $\mathcal{N}$ and $\mathcal{C}$. In our case this follows from the uniform mass ratio bounds on the underlying flow $\mathcal{M}$.

To compare the cylindrical tangent $\mathcal{C} \in T_{X_0}\mathcal{M}$ to a pseudo-tangent we first find a scale r at which $D_{X_0,r}\mathcal{M}$ is close to the cylindrical tangent at X_0, then use Lemma 8.3.1 to show $D_{X_0,r}\mathcal{M}$ is weakly close to $D_{Y,\lambda}\mathcal{M}$ for Y sufficiently close to X_0, and λ sufficiently close to r. This will show the pseudo-tangents $\mathcal{N}$ at points $Y \in S^+(X_0)$ close to X_0, and scales λ close to r will be shrinking cylinders by the rigidity theorem. We then proceed by an iterative argument making use of Lemma 8.3.1 to prove this for all sufficiently large λ.

Theorem 8.4.1. *Let $\mathcal{M}$ be an integral Brakke flow on an open subset $U \subset \mathbb{R}^{n+1}$ and for times $t \in [0, T)$. Suppose that $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$, with $t_0 > 0$, has a cylindrical tangent flow $\mathcal{C} \in T_X\mathcal{M}$ with spine dimension $k \geq 0$. Then there is $\delta = \delta(\mathcal{M}, X_0, n, k) > 0$ such that for any $\lambda > \delta^{-1}$ and $Y \in S^+(X_0) \cap P_\delta(X_0)$ any pseudo-tangent flow $\mathcal{N}$ at Y with scale λ is a shrinking cylinder with spine dimension k.*

Remark 8.4.2. In the proof we use that there is a sequence $r_j \to \infty$ such that $D_{X_0, r_j}\mathcal{M} \to \mathcal{C}$. In fact by virtue of the uniqueness result Theorem 7.4.9 any sequence $r_j \to \infty$ will have a subsequence such that $D_{X_0, r_j}\mathcal{M} \to \mathcal{C}$.

Proof. Let $\epsilon_* = \epsilon_*(\mathcal{M}, X_0, n, k) > 0$ denote the δ of Corollary 7.4.8. Note that necessary entropy bound for this result can be chosen depending only on the uniform mass ratio bounds of $\mathcal{M}$, see Definition 7.2.1. We first show $d(\mathcal{N}, \mathcal{C}) < \epsilon_*$ where $\mathcal{N}$ is a pseudo-tangent flow at $Y \in S^+(X_0) \cap P_{\lambda^{-1}}(X_0)$ with scale $\lambda > \delta^{-1}$ for some $\delta > 0$.

By definition of $\mathcal{C}$ as a tangent flow there is a sequence $r_j \to \infty$ such that $d(\mathcal{M}_{X_0, r_j}, \mathcal{C}) \to 0$. As such there is $J > 0$ such that

$$d(\mathcal{M}_{X_0, r_j}, \mathcal{C}) < \epsilon_*/3, \quad \text{for any } j \geq J. \tag{8.2}$$

Now let δ_1 denote the δ of the continuity of parabolic rescales result Lemma 8.3.1 with ϵ there replaced by $\epsilon_*/3$. We can find $J_1 \geq J$ such that $j \geq J_1$ implies $r_j > \delta_1^{-1}$. For any $j \geq J_1$ the result of Lemma 8.3.1 is that for any $\lambda \in [r_j/2, 2r_j]$ and $Y \in S^+(X_0) \cap P_{r_j^{-1}}(X_0)$ the following holds.

$$d(\mathcal{M}_{Y,\lambda}, \mathcal{M}_{X_0, r_j}) < \frac{\epsilon_*}{3}, \quad \text{for any } j \geq J_1. \tag{8.3}$$

Now let $R = R(\mathcal{M}, X_0, n, k)$ denote the r of the existence of pseudo-tangents result Corollary 8.2.6 with ϵ there replaced by $\epsilon_*/3$. We can choose $J_2 \geq J_1$ sufficiently large so that for each $j \geq J_2$ we have $r_j > 2/R$. Set $r = r_{J_2}$. Corollary 8.2.6 implies that for any pseudo-tangent $\mathcal{N}$ at $Y \in S^+(X_0) \cap P_{r^{-1}}(X_0)$ with scale $\lambda \geq r/2$ we have the following.

$$d(\mathcal{N}, \mathcal{M}_{Y,\lambda}) < \epsilon_*/3, \quad \text{for any } Y \in S^+(X_0) \cap P_{r^{-1}}(X_0),\ \lambda \geq \frac{r}{2}. \tag{8.4}$$

Setting $j = J_2$ in (8.2) (8.3), and combining these with (8.4) and the triangle inequality gives the following for $r = r(\mathcal{M}, X_0, n, k)$, where $\mathcal{N}$ is any pseudo-tangent at $Y \in S^+(X_0) \cap P_{r^{-1}}(X_0)$ with scale $\lambda \geq r/2$.

$$d(\mathcal{N},\mathcal{C}) < \epsilon_*, \quad \text{for any } Y \in S^+(X_0) \cap P_{r^{-1}}(X_0), \lambda \in [r/2, 2r]. \tag{8.5}$$

By Corollary 7.4.8 and (8.5) we have that the pseudo-tangents $\mathcal{N}$ are shrinking cylinders with spine dimension k for any $Y \in S^+(X_0) \cap P_{r^{-1}}(X_0)$ where $\lambda \in [r/2, 2r]$.

This is not the full result as it only covers the scales $\lambda \leq 2r$. To extend the result to any scale $\lambda > 2r$ we proceed by induction, making use of the continuity result Lemma 8.3.1. Let $r = r(\mathcal{M}, X_0, n, k) > 0$ be the radius we found so that (8.5) holds. Set $\delta = r^{-1}$ and $\delta_j = 2^{1-j}\delta$ for $j = 0, 1, \ldots$, and let $I_j = [(2\delta_j)^{-1}, 2\delta_j^{-1}]$. Then we can write

$$[(2\delta)^{-1}, \infty) = \bigcup_{j=0}^{\infty} I_j.$$

Note that the end point $2\delta_j^{-1}$ of I_j is equal to the midpoint δ_{j+1}^{-1} of I_{j+1}. The inductive hypothesis is that any pseudo-tangent at $Y \in S^+(X_0) \cap P_\delta(X_0)$ with scale $\lambda \in I_j$ is a shrinking cylinder with spine dimension k. In the $j = 0$ case this what was proved in (8.5). Now we assume the inductive hypothesis for $j = 0, 1, \ldots, p-1$.

Let $\mathcal{N}_1$ denote a pseudo-tangent at $Y \in S^+(X_0) \cap P_\delta(X_0)$ with scale $\lambda = \delta_p^{-1}$, and let $\mathcal{N}_2$ denote a pseudo tangent at Y with scale $\lambda \in I_p$. Since $\delta_p^{-1} = 2\delta_{p-1}^{-1} \in I_{p-1}$ we have that $\mathcal{N}_1$ is a shrinking cylinder with spine dimension k by the inductive hypothesis. As such by (8.4), Lemma 8.3.1 and the triangle inequality we have that

$$d(\mathcal{N}_1, \mathcal{N}_2) < \epsilon_*.$$

By the choice of ϵ_* this implies $\mathcal{N}_2$ is also a shrinking cylinders with spine dimension k. □

One of the key points of this is that there is a fixed radius $\delta > 0$ such that the pseudo-tangent $\mathcal{N}$ at a point $Y \in S^+(X_0) \cap P_\delta(X_0)$ is a shrinking cylinder for all sufficiently large scales $\lambda > \delta^{-1}$. However by Corollary 8.2.6 the pseudo-tangent $\mathcal{N}$ approximates $D_{Y,\lambda}\mathcal{M}$. Then for any sequence $\lambda_j \to \infty$ such that $D_{Y,\lambda_j}\mathcal{M}$ converges to some tangent flow $\mathcal{N}' \in T_Y\mathcal{M}$, there is sufficiently large j so that $D_{Y,\lambda_j}\mathcal{M}$ both well approximates $\mathcal{N}'$ and is well approximated by the pseudo-tangent $\mathcal{N}$, which is a shrinking cylinder. Choosing j sufficiently large so that we can apply the rigidity result Corollary 7.4.8 we can then show any tangent in $T_Y\mathcal{M}$ for $Y \in S^+(X_0) \cap P_\delta(X_0)$ is a shrinking cylinder.

Corollary 8.4.3. *Let $\mathcal{M}$ and $X_0 \in \mathrm{Sing}(\mathcal{M})$ be as in Theorem 8.4.1. Then there is $\delta(\mathcal{M}, X_0, n, k) > 0$ such that for any $Y \in S^+(X_0) \cap P_\delta(X_0)$, $T_Y\mathcal{M}$ contains a shrinking cylinder of spine dimension k.*

Remark 8.4.4. By the uniqueness Theorem 7.4.9 it follows that the shrinking cylinder is the unique tangent flow at such points Y.

Proof. Again take $\epsilon_* > 0$ to be the δ of Corollary 7.4.8. Choose $\delta > 0$ sufficiently small so that Theorem 8.4.1 applies with $\epsilon \le \epsilon_*$, and Corollary 8.2.6 applies on $S^+(X_0) \cap P_\delta(X_0)$ with ϵ.

As such $d(D_{Y,\lambda}\mathcal{M}, \mathcal{N}) < \epsilon$ for any pseudo-tangent $\mathcal{N}$ at $Y \in S^+(X_0) \cap P_\delta(X_0)$ and scale $\lambda > \delta^{-1}$. Further $\mathcal{N}$ is a shrinking cylinder with spine dimension k by Theorem 8.4.1. For any tangent $\mathcal{N}' \in T_Y\mathcal{M}$ we can choose λ_i so that $D_{Y,\lambda_i}\mathcal{M}$ converge to $\mathcal{N}'$. The class of shrinking cylinders with fixed spine dimension is compact by Proposition 7.4.3. As such we may find a subsequence of λ_i so that the pseudo-tangents $\mathcal{N}_i$ at Y with scale λ_i converge as Brakke flows to another shrinking cylinder $\mathcal{C}$ with spine dimension k. By Corollary 8.2.6 it follows that $d(\mathcal{C}, \mathcal{N}') \le \epsilon \le \epsilon_*$ and so $\mathcal{N}'$ is also a shrinking cylinder with spine dimension k.

□

By Corollary 8.4.3, once $\delta > 0$ is sufficiently small, $S = S^+(X_0) \cap \overline{P}_\delta(X_0)$ is a closed subset of cylindrical singularities. As such we can apply the method of Colding-Minicozzi [CM16] to acquire a structure result for $S^+(X_0) \cap P_\delta(X_0)$. The general idea is to use a parabolic Reifenberg theorem. The rescales $D_{Y,\lambda}\mathcal{M}$ at a cylindrical singularity Y eventually become graphical over the tangent flow, with a self-improving property. This makes it possible to approximate S by a collection of planes. Since S is compact we can find a finite set of points in S to approximate the whole set, which is why we can use a finite rather than countable union of Lipschitz submanifolds in both Colding-Minicozzi [CM16] and the following result.

Theorem 8.4.5 (Local Structure of $S^+(X_0)$)**.** *Let $\mathcal{M}$ be an n-dimensional integral Brakke flow for $t \in [0, T)$ and $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$ is a cylindrical singularity with spine dimension $k \ge 0$. Then there is some $\delta = \delta(n, k, \mathcal{M}, X_0)$ such that $\overline{P}_\delta(X_0) \cap S^+(X_0)$ is contained in a finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds and a parabolic $(n-2)$ dimensional set.*

Remark 8.4.6. Note that if we had a version of the David-Toro [DT12] Reifenberg Theorem 2.2.5 adapted to parabolic balls we could replace the finite union of parabolic $(n-1)$-dimensional Lipschitz submanifolds with a single parabolic $(n-1)$-dimensional submanifold that is the image of a Hölder continuous mapping of a disk. However it is not a simple rescaling argument due to the fact that a parabolic rescale by $\lambda > 0$ will rescale space by λ and time by λ^2.

8.5 The complement of $S^+(X_0)$

In the previous section we showed that local to a cylindrical singularity X_0, the subset $S^+(X_0) \subset \mathrm{Sing}(\mathcal{M})$ of the singular set is contained in a finite union of $(n-1)$-dimensional Lipschitz submanifolds, and an $(n-2)$-dimensional set.

We would like to extend this from $S^+(X_0)$ to $\mathrm{Sing}(\mathcal{M})$. We assume that the shrinking cylinder tangent at X_0 has spine dimension $n-1$. It suffices to show that local to a cylindrical singularity X_0, we have that $P_\delta(X_0) \cap \mathrm{Sing}(\mathcal{M}) \setminus S^+(X_0)$ is contained in a set with parabolic Hausdorff dimension $(n-2)$. By stratification Theorem 7.3.11 this would follow from ruling out certain tangent flows near X_0 with Gaussian density less than the Gaussian density of a shrinking cylinder with spine dimension $n-1$. By Proposition 7.2.27 the Gaussian density of a shrinking cylinder with spine dimension $n-1$ is $\sqrt{2\pi/e}$.

For a general integral Brakke flow a difficulty arises. Triple junction singularities are part of the parabolic $(n+1)$-dimensional stratum, and have Gaussian density $3/2 < \sqrt{2\pi/e}$, so could occur as tangents at points outside of $S^+(X_0)$. These triple junction singularities could occur very close to a cylindrical singularity X_0. Another problematic tangent is the homothetic Brakke spoons which can occur at singularities in the parabolic $(n-1)$-dimensional stratum.

However it is possible to identify some subclasses of integral Brakke flows for which we can control the nearby low density singularities. One such class is the class of mean-convex flows. It was shown by White [Whi03] that mean convex flows only develop cylindrical singularities. In this case if $\mathcal{M}$ is a mean-convex integral n-dimensional Brakke flow and X_0 a cylindrical singularity with spine dimension $n-1$, then $S^+(X_0)$ is exactly the top strata of singularities $\mathrm{Sing}_{n-1}(\mathcal{M})$, and Theorem 8.4.5 is exactly what was proved by Colding-Minicozzi [CM16].

Another class of integral Brakke flows are those that arise from Ilmanen's elliptic regularisation procedure [Ilm94]. If $\mathcal{M}$ is such a Brakke flow, then $\mathcal{M} \times \mathbb{R}$ is approximated by a sequence of smooth hypersurfaces. As such any closed curve that intersects $\mathrm{Reg}(\mathcal{M})$ transversally will eventually intersect the approximating smooth hypersurfaces transversally due to Brakke's regularity Theorem 7.2.38. This curve must intersect the smooth hypersurfaces in an even number of points, and as such intersects $\mathrm{Reg}(\mathcal{M})$ in an even number of points. Using this we can rule out triple junctions and Brakke spoons occurring as tangents to a Brakke flow arising from elliptic regularisation.

One way to think about this is that the triple junction is non-orientable. Ilmanen

[Ilm94] shows that flows arising from elliptic regularisation have an associated flow of currents, which are orientable by definition.

In fact this parity property that a closed curve intersects $\mathrm{Reg}(\mathcal{M})$ transversely in an even number of points can be used to define an abstract closed class of flows $\mathcal{C}$, which includes any unit-regular flow arising from elliptic regularisation.

The elliptic regularisation procedure applies to currents. We will briefly introduce currents here, as in section 1 of Ilmanen [Ilm94].

Definition 8.5.1 (Currents)**.** Let $\Lambda_k U$ and $\Lambda^k U$ denote the space of alternating k-vectors and alternating k-forms on $U \subset \mathbb{R}^n$ respectively. A k-current is a continuous linear functional T of the space of smooth compactly supported k-forms $C_c^\infty(\Lambda^k U)$.

The mass measure of a current T on U is

$$\nu_T(V) = \sup\left\{T(\phi) \,:\, \phi \in C_c^\infty(\Lambda^k U \lfloor V),\, |\phi| \leq 1\right\}.$$

The mass of T is $\mathbf{M}(T) = \nu_T(U)$.

As with varifolds, we generally work with currents with more structure. First we need to define an orientation to integer rectifiable Radon measures.

Definition 8.5.2 (Orientation)**.** Let μ denote an integer k-rectifiable Radon measure supported on $M \subset U \subset \mathbb{R}^n$. We say a μ-measurable section ξ of $\Lambda_k U\lfloor M$ is an orientation of μ if $\xi(x)$ is an orthonormal basis of the approximate tangent space $T_x\mu$ for μ-almost every $x \in U$.

A current is locally integer rectifiable when it can be defined by an integer rectifiable Radon measure with an orientation.

Definition 8.5.3 (Locally Integer Rectifiable Currents)**.** Let T be a k-current on U. Suppose there is an integer k-rectifiable Radon measure μ with support $M \subset U$ and an orientation ξ. Further suppose

$$T(\phi) = \int_M \langle \phi, \xi \rangle \,\mathrm{d}\mu, \quad \phi \in C_c^\infty(\Lambda^k U).$$

Then we say T is a locally integer rectifiable k-current.

Definition 8.5.4 (Boundary Operator)**.** Given a k-current T on U we define the boundary ∂T, a $(k-1)$-current on U, by

$$\partial T(\phi) = T(\mathrm{d}\phi), \quad \phi \in C_c^\infty(\Lambda^k U).$$

Here $\mathrm{d}\phi$ denotes the exterior derivative of the alternating k-form ϕ.

Remark 8.5.5. If T is locally integer rectifiable and ∂T has locally finite mass, then ∂T is also locally integer rectifiable, for example see Simon [Sim83b] Theorem 30.3.

Currents admit a weak topology as follows.

Definition 8.5.6 (Weak Topology of Currents)**.** Given k-currents T_i and a k-current T on $U \subset \mathbb{R}^n$ we say $T_i \to T$ weakly if $T_i(\phi) \to T(\phi)$ for all $\phi \in C_c^\infty(\Lambda^k U)$.

We will need to be able to slice currents. We can then compare the time slices of a spacetime current to the Radon measures in a Brakke flow.

Definition 8.5.7 (Slices of currents)**.** Let T denote a locally integer rectifiable $(k+1)$-current on $U \times \mathbb{R}$ for $U \subset \mathbb{R}^n$, and suppose ∂T is locally integer rectifiable. Then for each $t \in \mathbb{R}$ we define

$$T_t = \partial(T \lfloor U \times [t, \infty)).$$

For more details on currents see section 1 of Ilmanen [Ilm94] and chapter 6 of Simon [Sim83b].

Below we briefly outline the elliptic regularisation procedure and prove that for a flow arising from this procedure $\mathrm{Sing}(\mathcal{M}) \setminus S^+(X_0)$ is at most parabolic Hausdorff dimension $n-2$ when X_0 is a cylindrical singularity with spine dimension $n-1$.

Theorem 8.5.8 (Elliptic Regularisation)**.** *Let T_0 be a k-dimensional locally integral current in $\mathbb{R}^{n+1}$, with $\partial T_0 = 0$ and $\mathbf{M}(T_0) < \infty$. Then there is a locally integral $(k+1)$-dimensional current T on $\mathbb{R}^{n+1} \times [0, \infty)$ and a family of Radon measures $\{\mu_t\}_{t \geq 0}$ such that the following holds. Let ν_{T_t} denote the mass measure of the slice T_t. Then we have that*

$$\partial T = T_0, \mu_0 = \nu_{T_0}, \mu_t \geq \nu_{T_t} \text{ for } t \geq 0.$$

Further $\mathbf{M}(T_B)$ is absolutely continuous with respect to $\mathcal{L}^1(B)$ for $B \subset \mathbb{R}$, $\mathbf{M}(\mu_t) \leq \mathbf{M}(\mu_0)$ for all $t > 0$, and the Radon measures $\{\mu_t\}_{t \geq 0}$ form a Brakke flow.

As noted in 8.2 of Ilmanen [Ilm94], the mass of the Brakke flow μ_t cannot suddenly disappear entirely, unless the current T_t also disappears. That is suppose $\mu_t = 0$ for all $t > t_0$. Then $T_t = 0$ for all $t > t_0$ by the fact that $\mu_t \geq \nu_{T_t}$. We have that T_t is continuous with respect to t in the weak topology of currents by the absolute continuity of $\mathbf{M}(T_B)$ with respect to $\mathcal{L}^1(B)$ for $B \subset \mathbb{R}$. Then $T_t = 0$ for $t > t_0$ implies $T_{t_0} = 0$ also. Evidently if $T_0 \neq 0$ this implies t_0 cannot be 0, that is mass cannot instantly disappear. In fact as long as the current hasn't disappeared, the

Brakke flow μ_t cannot disappear instantaneously, in particular implying μ_t cannot admit quasi-static unit density planes as tangent flows until the current disappears.

One of the ideas behind elliptic regularisation is to approximate the parabolic mean curvature flow by surfaces satisfying an elliptic partial differential equation. This can be done by studying the translative functional.

Lemma 8.5.9 (Translative Functional)**.** *Let e_{n+1} denote the $(n+1)$-th standard basis vector in $\mathbb{R}^{n+1}$ and let $z : \mathbb{R}^{n+1} \to \mathbb{R}$ denote the function $z(x) = x \cdot e_{n+1}$. Given a surface $M \subset \mathbb{R}^{n+1}$ and $\lambda > 0$ define*

$$I_\lambda(M) = \int_M \exp(-\lambda z)\,\mathrm{d}A. \tag{8.6}$$

Then M is a critical point of I_λ if and only if $M - \lambda t e_{n+1}$ is a mean curvature flow.

Remark 8.5.10. In other words the mean curvature flows that move by translating a surface along the $(n+1)$-th dimension with speed λ are exactly the critical points of I_λ. As such I_λ is called the translative functional.

Now given a compact surface $\Sigma \subset \mathbb{R}^{n+1}$ we look for minimisers $M_\lambda \subset \mathbb{R}^{n+2}$ of I_λ subject to $\partial M_\lambda = \Sigma$. Under certain conditions, in the limit as $\lambda \to \infty$, these M_λ will approximate a Brakke flow starting from Σ.

An n-dimensional Brakke flow $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}^{n+1}$ arising from this procedure can be extended trivially to $\mathbb{R}^{n+2}$. This extension can then be approximated by hypersurfaces in $\mathbb{R}^{n+2}$ which are singular on a set of codimension at least 6. The following parity property is discussed in Chodosh's notes from White's lectures [CW15].

Proposition 8.5.11 (Parity)**.** *Let $\mathcal{M}$ be an n-dimensional Brakke flow arising from elliptic regularisation for $n = 2, \ldots, 5$. Suppose γ is a closed curve in spacetime $\mathbb{R}^{n+1,1}$ intersecting $\mathrm{Reg}(\mathcal{M})$ transversely, and never meeting $\mathrm{Sing}(\mathcal{M})$. Then γ intersects $\mathrm{Reg}(\mathcal{M})$ in an even number of points.*

Remark 8.5.12. For $n \geq 6$, the approximating surfaces of elliptic regularisation are at least 7-dimensional surfaces that are stationary for the translative functional (8.6). As such the tangent cones to the approximating surfaces are also at least 7-dimensional, and may admit singularities. Singularities can be ruled out when $n+1 \leq 6$ by Schoen-Simon-Yau [SSY75], for example see Theorem B.2 in the appendix of Simon's notes [Sim83b].

The result can be extended to the case $n \geq 6$ under the assumption that $\gamma = \partial D$ is the boundary of a flat 2-dimensional disk $D \subset \mathbb{R}^{n+1} \times \{t\}$ intersecting $\mathrm{Reg}(\mathcal{M}_t)$

transversely, similar to Proposition 4.3 of Schulze-White [SW17]. The proof in this case uses that the singular set of the approximating surfaces has parabolic Hausdorff codimension at least 6. The spacetime orthogonal complement $D^\perp$ has parabolic Hausdorff dimension $n+1$, and as such we can find a small vector $v \in D^\perp$ to perturb D so that $D + v$ is disjoint from the singular set of the approximating surface.

Proof. This follows from Brakke regularity Theorem 7.2.38. By definition of elliptic regularisation the product of $\mathcal{M}$ with $\mathbb{R}$ can be approximated by smooth hypersurfaces $\mathcal{M}_i$ that are stationary for the translative functional (8.6). For sufficiently large i, γ interesects $\mathcal{M}$ transversely in a regular point if and only if γ intersects $\mathcal{M}_i$ transversely in a regular point. This follows since $\mathcal{M}_i$ converge smoothly to $\mathcal{M}$ local to points $x \in \mathrm{Reg}(\mathcal{M})$ by Brakke regularity Theorem 7.2.38.

Since γ can only intersect such a smooth hypersurface transversely in an even number of regular points it follows that it can only intersect $\mathrm{Reg}(\mathcal{M})$ transversely in an even number of points. □

This parity property can be used to rule out triple junction type singularities. Proposition 8.5.11 shows that low dimensional flows arising from elliptic regularisation have this parity property. We will show that the set of all flows that have this parity property form a closed class, as in Theorem 13.1 of Chodosh's notes from White's lectures [CW15].

We also need to rule out quasi-static unit density plane type singularities. A point $X \in \mathrm{Sing}(\mathcal{M})$ at which there is a quasi-static unit density plane has Gaussian density $\Theta_{\mathcal{M}}(X) = 1$. The class of unit regular flows do not admit unit-density quasi-static tangent planes. This class was studied in Schulze-White [SW17].

Definition 8.5.13 (Unit Regular Class)**.** We say an integral Brakke flow $\mathcal{M}$ is unit regular if $\Theta_{\mathcal{M}}(X) = 1$ implies $X \in \mathrm{Reg}(\mathcal{M})$.

We can then define an abstract class of unit regular flows $\mathcal{C}$ such that a parity property as in Proposition 8.5.11 holds. We can show such a class is closed under limits of Brakke flows.

Definition 8.5.14 (Even parity flows)**.** Let $\mathcal{C}$ denote the class of unit regular integral Brakke flows $\mathcal{M}$ such that any closed spacetime curve γ in $\mathbb{R}^{n+1,1} \setminus \mathrm{Sing}(\mathcal{M})$ intersects $\mathrm{Reg}(\mathcal{M})$ transversely in an even number of points.

Remark 8.5.15. Clearly any unit-regular n-dimensional Brakke flow arising from elliptic regularisation for $n = 2, \ldots, 5$ is in the class $\mathcal{C}$ by Proposition 8.5.11.

It is clear that $\mathcal{C}$ is closed under parabolic rescaling. We also have that $\mathcal{C}$ is closed under limits of Brakke flows.

Proposition 8.5.16 ($\mathcal{C}$ is closed)**.** *Let $\mathcal{M}_i \in \mathcal{C}$ and $\mathcal{M}_i \to \mathcal{M}$, then $\mathcal{M} \in \mathcal{C}$.*

Proof. The proof that $\mathcal{M}$ is unit regular can be found in Schulze-White [SW17], and is as follows. Suppose $X_i \in \mathcal{M}_i$ converge to X and $\Theta_{\mathcal{M}}(X) = 1$. Then $\Theta_{\mathcal{M}_i}(X_i) \to 1$ by upper-semicontinuity. Let $\mathcal{N}_i \in T_{X_i}\mathcal{M}_i$, then $\Theta_{\mathcal{N}_i}(0) \to 1$. Eventually $\Theta_{\mathcal{N}_i}(0) < 1+\epsilon$ where $\epsilon > 0$ is from Allard's regularity theorem [All72]. Since the $t = -1$ slices of $\mathcal{N}_i$ are stationary with respect to a Gaussian weighted metric, the regularity theorem implies these slices are smooth, implying $\mathcal{N}_i$ are smooth at all times $t < 0$. Then White's regularity [Whi05], with the fact that $\Theta_{\mathcal{N}_i}(0) < 2$ and that $\mathcal{M}_i$ are unit regular implies $\mathcal{N}_i$ are unit density planes. As such by Brakke regularity [Bra78] we have $X_i \in \mathrm{Reg}(\mathcal{M}_i)$. This implies $X \in \mathrm{Reg}(\mathcal{M})$ since X_i was arbitrary, implying there is a neighbourhood $U = U(\epsilon)$ of X on which $\mathcal{M}_i$ are smooth with Gaussian densities $\leq 1+\epsilon$ for any $\epsilon > 0$. Then White's local regularity [Whi05] implies $\mathcal{M}$ is smooth on U, implying $X \in \mathrm{Reg}(\mathcal{M})$.

The parity property follows by Brakke regularity [Bra78]. Indeed if γ is a curve in spacetime intersecting $\mathrm{Reg}(\mathcal{M})$ transversely, then for sufficiently large i, by Brakke regularity [Bra78] we have that γ intersects $\mathrm{Reg}(\mathcal{M}_i)$ transversely, and by definition of $\mathcal{C}$ this occurs at an even number of points. $\square$

For the class $\mathcal{C}$ we can rule out both unit-density quasi-static planes and triple junctions as tangents. This allows us to clear out the low density top dimensional singular strata. In the following $\mathcal{M} \in \mathcal{C}$ is a fixed n-dimensional integral Brakke flow in the class. The following arguments are briefly outlined in the proof of Theorem 13.2 from Chodosh's lecture notes on White's lectures [CW15].

Lemma 8.5.17 ($\mathrm{Sing}_{n+2}(\mathcal{M})$)**.** *There are no singularities in $\mathrm{Sing}_{n+2}(\mathcal{M})$ with Gaussian density less than 2.*

Remark 8.5.18. This is true for any integral Brakke flow.

Proof. By White's [Whi97] stratification Theorem 7.3.11 we know that a singularity $Y \in \mathrm{Sing}_{n+2}(\mathcal{M})$ must have a tangent flow that is a static stationary n-dimensional cone with n-dimensional spine. Clearly these can only be n-dimensional planes with multiplicity. A multiplicity 1 plane cannot occur as a tangent flow to a point $Y \in \mathrm{Sing}(\mathcal{M})$ by Brakke regularity Theorem 7.2.38. As such the Gaussian density of Y is at least 2, proving the result. $\square$

Lemma 8.5.19 ($\mathrm{Sing}_{n+1}(\mathcal{M})$)**.** *There are no singularities in* $\mathrm{Sing}_{n+1}(\mathcal{M})$ *with Gaussian density less than* 2.

Remark 8.5.20. This only requires the even parity property of flows in $\mathcal{C}$.

Proof. The $(n+1)$-th singular strata contains singularities $Y \in \mathrm{Sing}(\mathcal{M})$ at which there is a tangent flow that is a static stationary n-dimensional cone with an $(n-1)$-dimensional spine. By taking a cross-section along the spine it is clear that all such cones are unions of half planes meeting along their boundary in equal angles. Clearly there need to be at least 3 half planes. The Gaussian density of k half planes arranged in this manner is $k/2$. As such only the case of 3 half planes, a triple junction, can occur as a tangent to a point $Y \in \mathrm{Sing}(\mathcal{M})$ with Gaussian density less than 2. However this is ruled out by the even parity of flows $\mathcal{M} \in \mathcal{C}$, since we can construct a curve that intersects the regular part of the triple junction transversely in 3 points, and this curve would intersect $\mathcal{M}$ in 3 points also due to Brakke regularity and the fact that $D_{Y,\lambda_i}\mathcal{M}$ converge to the triple junction for some $\lambda_i \to \infty$. □

Lemma 8.5.21 ($\mathrm{Sing}_n(\mathcal{M})$)**.** *There are no singularities in* $\mathrm{Sing}_n(\mathcal{M})$ *with Gaussian density less than* 2.

Proof. The singularities $Y \in \mathrm{Sing}_n(\mathcal{M})$ must have a tangent flow in one of two forms. Either there is a quasi-static n-dimensional stationary cone with n-dimensional spine as a tangent flow to Y, or there is a static n-dimensional stationary cone with $(n-2)$-dimensional spine as a tangent flow at Y.

A unit-density quasi-static plane cannot occur as a tangent to a unit-regular flow $\mathcal{M} \in \mathcal{C}$, so the former case can only happen at points Y with Gaussian density at least 2.

In the latter case we have a stationary cone with $(n-2)$-dimensional spine. By taking the cross-section along the spine of this cone we acquire a stationary 2 dimensional cone $\Sigma \subset \mathbb{R}^3$. The link is the intersection of Σ with S^2, that is $\hat{\Sigma} = \Sigma \cap S^2$, and it is well known that this forms a geodesic network on S^2. Assuming such a cone arises as a tangent flow to a point Y with Gaussian density less than 2, it follows that the density at all points of this geodesic network are also less than 2. As such the only possible singularity on the link $\hat{\Sigma}$ is a triple junction of geodesics, which can be ruled out by the parity property of $\mathcal{C}$. So we must have that $\hat{\Sigma}$ is smooth, implying it is a great circle on S^2, which ultimately implies the original tangent flow at Y we a unit density plane, which cannot occur as a tangent flow at a singular point due to Brakke regularity. □

Lemma 8.5.22 ($\mathrm{Sing}_{n-1}(\mathcal{M})$)**.** *A singularity $Y \in \mathrm{Sing}_{n-1}(\mathcal{M})$ with Gaussian density less than 2 admits a tangent flow that is either a shrinking cylinder with $(n-1)$-dimensional spine, or a static stationary n-dimensional cone with $(n-3)$-dimensional spine such that the cross-section along the spine has an isolated singularity at the origin.*

Proof. By stratification any singularity $Y \in \mathrm{Sing}_{n-1}(\mathcal{M})$ has a tangent flow that is an n-dimensional static stationary cone with $(n-3)$-dimensional spine, an n-dimensional quasi-static stationary cone with $(n-1)$-dimensional spine, or an n-dimensional non-stationary self-shrinking Brakke flow with $(n-1)$-dimensional spine.

In the first case let $\Sigma \subset \mathbb{R}^4$ denote the stationary 3-dimensional cross-section of the cone along the spine. The aim is to show Σ is smooth away from the origin. As Σ is a cone it suffices to show the link $\hat{\Sigma} = S^3 \cap \Sigma$ is smooth. The link $\hat{\Sigma}$ is a 2-dimensional minimal surface on S^3. Assuming Σ arises as a the cross-section of a tangent to a point Y with Gaussian density less than two, it follows that any singularity of $\hat{\Sigma}$ has density less than 2. Now supposing $x \in \mathrm{Sing}(\hat{\Sigma})$, a tangent cone $C \in T_x\hat{\Sigma}$ at this singularity would split off another dimension, and as such C is a 2-dimensional stationary cone with spine dimension 1 or 2, and density at the origin $\Theta_C(0) < 2$. The only possibility is that this cone is a triple junction. So any singularity on $\hat{\Sigma}$ is a triple junction. However we can rule out triple junctions again by the parity of $\mathcal{C}$. As such $\hat{\Sigma}$ is smooth.

The second case of a quasi-static n-dimensional stationary cone with $(n-1)$-dimensional spine can only be a quasi-static triple junction by an argument similar to Lemma 8.5.19. However quasi-static triple junctions can also be ruled out by the parity of $\mathcal{C}$.

The final case is that there is a non-stationary self-shrinking Brakke flow with $(n-1)$-dimensional spine as a tangent flow at Y. The shrinking cylinder is one example. The cross-section along the spine in this case is a self-shrinking planar curve. By the parity of $\mathcal{C}$, any self-intersection of this curve has a tangent cone that is the union of an even number of rays. The case of 2 rays is regular, so the density any singularity on the self-shrinking curve is at least the density of 4 rays, that is at least 2. As such the shrinking cylinder is the only example with Gaussian density less than 2. $\square$

Finally we only have to rule out the case of a singularity $Y \in \mathrm{Sing}(\mathcal{M})$ with Gaussian density less than $\sqrt{2\pi/e}$ and at which there is a tangent flow that is an n-dimensional stationary cone with $(n-3)$-dimensional spine. It suffices to show that a smooth minimal surface on S^3 has an area lower bound, implying the cone over this surface has a density lower bound of $\sqrt{2\pi/e}$. To this end we can make use of the following two results. Firstly a classification of genus 0 smooth minimal surfaces on S^3, due to Almgren [Alm66].

Theorem 8.5.23 (Classification of genus 0 minimal surfaces on S^3)**.** *A smooth embedded minimal surface on S^3 with genus 0 is a rotation of the equator $S^2 \times \{0\}$.*

Let $\hat{\Sigma} \subset S^3$ denote a smooth minimal surface on S^3 obtained from a tangent to a singularity $Y \in \mathrm{Sing}_{n-1}(\mathcal{M})$. Theorem 8.5.23 shows that the genus of $\hat{\Sigma}$ must be at least 1, otherwise $\hat{\Sigma}$ is the equator, which would imply the tangent flow at Y is simply a unit-density plane, which cannot happen due to Brakke regularity Theorem 7.2.38.

Next we have the following area lower bound for minimal surfaces on S^3 with genus at least 1. This result is due to Marques-Neves [MN14] and proved using a min-max procedure. It shows that the Clifford torus attains the minimal area amongst minimal surfaces on S^3 with positive genus.

Theorem 8.5.24 (Area Lower Bounds)**.** *Let $\hat{\Sigma} \subset S^3$ be a smooth minimal surface with genus at least 1. Then $\hat{\Sigma}$ has area at least $2\pi^2$.*

Now suppose $\hat{\Sigma} \subset S^3$ is a smooth minimal surface on S^3 with genus at least 1. By Theorem 8.5.24 we can compute a lower bound for the density of the cone $\Sigma \subset \mathbb{R}^4$ over $\hat{\Sigma}$ as follows.

$$\Theta_\Sigma(0) = \frac{\mathcal{H}^3(\Sigma \cap B_1)}{\omega_3} = \frac{\int_0^1 \mathcal{H}^2(\Sigma \cap S^3_r)\,\mathrm{d}r}{\omega_3}.$$

Since Σ is a cone, the area lower bound of $2\pi^2$ for $\hat{\Sigma}$ gives that $\mathcal{H}^2(\Sigma \cap S^3_r) \geq 2\pi^2 r^2$. As such since $\omega_3 = \frac{4}{3}\pi$ we have that

$$\Theta_\Sigma(0) \geq \left(\frac{4}{3}\pi\right)^{-1} \int_0^1 2\pi^2 r^2\,\mathrm{d}r = \frac{\pi}{2}.$$

One can compute that $\frac{\pi}{2} > \sqrt{2\pi/e}$, which proves the following extension of Lemma 8.5.22.

Corollary 8.5.25 ($\mathrm{Sing}_{n-1}(\mathcal{M})$)**.** *The only singularities in $\mathrm{Sing}_{n-1}(\mathcal{M})$ with Gaussian density at most equal to $\sqrt{2\pi/e}$ are the cylindrical singularities admitting a tangent flow that is a shrinking cylinder with $(n-1)$-dimensional spine.*

Now if $X_0 \in \mathrm{Sing}(\mathcal{M})$ is a cylindrical singularity which admits a tangent flow that is a shrinking cylinder with $(n-1)$-dimensional spine, then we have that $\mathrm{Sing}(\mathcal{M}) \setminus$

$S^+(X_0)$ is a collection of singularities with Gaussian density less than $\sqrt{2\pi/e}$. As such we have the following by combining Lemmas 8.5.17, 8.5.19, 8.5.21, 8.5.22 and Corollary 8.5.25.

Lemma 8.5.26 (Lower Dimensionality of $\mathrm{Sing}(\mathcal{M})\setminus S^+(X_0)$)**.** *Let $\mathcal{M} \in \mathcal{C}$ be a Brakke flow and $X_0 \in \mathrm{Sing}(\mathcal{M})$ a cylindrical singularity with spine dimension $(n-1)$. Then $\mathrm{Sing}(\mathcal{M}) \setminus S^+(X_0)$ is at most parabolic Hausdorff dimension $n-2$.*

Proof. This follows immediately from the stratification Theorem 7.3.11 and the fact that the Gaussian density is bounded above by $\sqrt{2\pi/e} < 2$ on $\mathrm{Sing}(\mathcal{M}) \setminus S^+(X_0)$. □

We can now extend Theorem 8.4.5 to this class of flows arising from elliptic regularisation.

Theorem 8.5.27 (Structure of $\mathrm{Sing}(\mathcal{M})$)**.** *Let $\mathcal{M} \in \mathcal{C}$ denote a Brakke flow in $\mathbb{R}^{n+1}$ for times $t \in [0,T)$, and suppose $X_0 = (x_0, t_0) \in \mathrm{Sing}(\mathcal{M})$ with $t_0 > 0$ is a cylindrical singularity with spine dimension $(n-1)$. Then there is $\delta = \delta(\mathcal{M}, X_0, n) > 0$ such that $\mathrm{Sing}(\mathcal{M}) \cap P_\delta(X_0)$ is contained in the union of a finite union of parabolic Hausdorff dimension $(n-1)$ Lipschitz submanifolds of $\mathbb{R}^{n+1}$, and a set with parabolic Hausdorff dimension at most $(n-2)$.*

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